A PROCESS CAPABILITY INDEX FOR AVERAGES OF OBSERVATIONS FROM NEW BATCHES IN THE CASE OF THE BALANCED RANDOM EFFECTS MODEL WITH THREE VARIANCE COMPONENTS

1. INTRODUCTION

Data arising from multiple sources of variability are very common in practice. Virtually all industrial processes exhibit between-batch, as well as within-batch components of variation. In some cases the between-batch (or between subgroup) component is viewed as part of the common-cause-system for the process. It seems worthwhile to develop process capability indices in more general settings. In this paper, a three variance components model is considered.

A version of the process capability or performance index for the balanced random effects model with three variance components from a Bayesian framework is considered. The index is denoted as ${}_3P_{pl}^{\,1}$ and can be used for average of observations for a given time-period.

2. DEFINITIONS AND NOTATIONS

The lower process performance index for the three variance component model is defined as:

$$ {}_3P_{pl}^1 = \frac{\mu - l_0}{3\left(\frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{JK}\right)^{\frac{1}{2}}} \qquad (2.1) $$

where

$\mu = \text{mean of future observation for a new or unknown batch}$

$\sigma_1^2 = \text{residual variance}$

$\sigma_2^2 = \text{within groups variance}$

$\sigma_3^2 = \text{between groups variance}$

$l_0 = \text{lower specification limit}$

$J = \text{batch (package) size}$

$K = \text{number of sub-samples per batch (package)}$

The interest is whether or not a process is capable of producing to a specification of least l_0 over say a one month future period (I) (see example).

As introduced above, the variation observed could possibly be explained by several components such as a "*between*" months (month-to-month) component, a "*within*" months (package- to-package) component and a *residual* component. These sources of variation should be incorporated in a suitable model.

Approximations of the exact posterior distribution of ${}_3P_{pl}^{1}$ can be obtained. Current knowledge indicates that a posterior analysis for this form of capability index does not exist.

The above mentioned index will be contrasted with the following indices:

i. $${}_3P_{pl}^{11} = \left(\frac{\mu - l_0}{3\sigma_{total}}\right) = \left(\frac{\mu - l_0}{3\left(\sigma_1^{\,2} + \sigma_2^{\,2} + \sigma_3^{\,2}\right)^{\frac{1}{2}}}\right) \tag{2.2}$$

which is independent of I, J and K. This index assesses whether the process is capable of producing **each** future tablet to specification.

ii. $${}_3P_{pl}^{\,111} = \left(\frac{\mu - l_0}{3\left(\frac{\sigma_1^{\,2} + K\sigma_2^{\,2} + K\sigma_3^{\,2}}{K}\right)^{\frac{1}{2}}}\right) \tag{2.3}$$

which assesses whether the process is capable of producing a future **batch** to specification**.**

iii. $${}_3P_{pl}^{1V} = \frac{\mu - l_0}{3\left(\frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{IJK}\right)^{\frac{1}{2}}} \tag{2.4}$$

which assesses whether the process is capable of producing to specification over a **future period** of I months.

iv. $${}_3P_{pl}^{V} = \left(\frac{\mu_i - l_0}{3\left(\frac{\sigma_1^{\,2}}{JK} + \frac{\sigma_2^{\,2}}{J}\right)^{\frac{1}{2}}}\right) = \left(\frac{\mu_i - l_0}{3\left(\frac{\sigma_1^{\,2} + K\sigma_2^{\,2}}{JK}\right)^{\frac{1}{2}}}\right) \quad \mu_i = \mu + r_i\,. \tag{2.5}$$

which is independent $\sigma_3^{\,2}$ and assesses whether the process is capable of producing to specification **in a specific month**, say the 10th month or a month similar to the 10th month.

In the next section we will look at the three variance component model. The posterior distribution of the mean and variance components is derived in section 4. In section 5, the posterior distribution of ${}_3P_{pl}^{\,1}$ conditional on the variance components is derived. The probability–matching prior for ${}_3P_{pl}^{\,1}$ will be derived in section 6. Sections 7 and 8 deal with the estimation of the indices. We conclude with an application in section 9.

3. **THE VARIANCE COMPONENT MODEL**

The variance component model with three variance components is of the form:

$$Y_{ijk} = \mu + r_i + c_{ij} + \varepsilon_{ijk} \quad i = 1,\ldots,I, \quad j = 1,\ldots,J \quad \text{and} \quad k = 1,\ldots,K. \tag{3.1}$$

where Y_{ijk} are the observations, μ is a common location parameter, r_i, c_{ij} and ε_{ijk} are three different kinds of random effects. We further assume that the random effects $(r_i, c_{ij}, \varepsilon_{ijk})$ are all independent and that

$$r_i \sim N(0,\sigma_3^2), \quad c_{ij} \sim N(0,\sigma_2^2) \quad \text{and} \quad \varepsilon_{ijk} \sim N(0,\sigma_1^2).$$

and the parameters $(\sigma_1^2, \sigma_2^2, \sigma_3^2)$ are the variance components. This model explains each data point as being additively influenced by three random effects (*residual, batches (packages)* and *groups)*. The variance of each data point therefore consists of three components corresponding to each of these random effects.

The variance components for our earlier example consist of *residual, packages* and *months* effects and are then denoted by $(\sigma_1^2, \sigma_2^2, \sigma_3^2)$. Some important results of the variance component model are summarized in theorem 3.1.

Theorem 3.1

I. $Y_{ijk} \mid \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N(\mu, \sigma_1^2 + \sigma_2^2 + \sigma_3^2)$

II. $\overline{Y_{ij.}} \mid \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N(\mu, \sigma_3^2 + \sigma_2^2 + \frac{\sigma_1^2}{K}) = N(\mu, \frac{K\sigma_3^2 + K\sigma_2^2 + \sigma_1^2}{K})$

III. $\overline{\overline{Y_{i..}}} \mid \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N(\mu, \sigma_3^2 + \frac{\sigma_2^2}{J} + \frac{\sigma_1^2}{KJ}) = N(\mu, \frac{JK\sigma_3^2 + K\sigma_2^2 + \sigma_1^2}{JK})$

IV. $\overline{\overline{\overline{Y_{...}}}} \mid \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N(\mu, \frac{\sigma_3^2}{I} + \frac{\sigma_2^2}{IJ} + \frac{\sigma_1^2}{IJK}) = N(\mu, \frac{JK\sigma_3^2 + K\sigma_2^2 + \sigma_1^2}{IJK})$

V. $\overline{\overline{Y_{i..}}} \mid \theta, r_i, \sigma_1^2, \sigma_2^2 \sim N(\theta + r_i, \frac{\sigma_2^2}{J} + \frac{\sigma_1^2}{KJ}) = N(\theta + r_i, \frac{K\sigma_2^2 + \sigma_1^2}{JK})$

Proof

The proofs are very simple and similar and only proof to III and V are given in the Appendix.

4. POSTERIOR DISTRIBUTION OF THE MEAN AND THE VARIANCE COMPONENTS

Consider the model in equation 3.1 again:

$$Y_{ijk} = \mu + r_i + c_{ij} + \varepsilon_{ijk} \quad i = 1,\ldots,I, \quad j = 1,\ldots,J \quad \text{and} \quad k = 1,\ldots,K.$$

where

$$r_i \sim N(0,\sigma_3^2), \quad c_{ij} \sim N(0,\sigma_2^2) \quad \text{and} \quad \varepsilon_{ijk} \sim N(0,\sigma_1^2).$$

Theorem 4.1

For the non-informative joint prior (see Box and Tao (1973)):

$$p(\mu,\sigma_1^2,\sigma_{12}^2,\sigma_{123}^2) \propto p(\mu)\,p(\sigma_1^2,\sigma_{12}^2,\sigma_{123})$$

$$= c \times p(\sigma_1^2,\sigma_{12}^2,\sigma_{123})$$

$$\propto \sigma_1^{-2}(\sigma_1^2 + K\sigma_2^2)^{-1}(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^{-1}$$

The joint posterior distribution of σ_1^2, σ_2^2 and σ_3^2 can be worked out as

$$p\left(\sigma_1^2,\sigma_{12}^2,\sigma_{123}^2 \mid \underline{Y}\right) \propto (\sigma_1^2)^{\frac{-(\frac{1}{2}v_1+2)}{2}} \exp\left(-\frac{1}{2}\left\{\frac{v_1 m_1}{\sigma_1^2}\right\}\right) \times (\sigma_1^2 + K\sigma_2^2)^{\frac{-\frac{1}{2}(v_2+2)}{2}} \exp\left(-\frac{1}{2}\left\{\frac{v_2 m_2}{(\sigma_1^2 + K\sigma_2^2)}\right\}\right) \times$$

$$(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^{\frac{-\frac{1}{2}(v_3+2)}{2}} \exp\left(-\frac{1}{2}\left\{\frac{v_3 m_3}{(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)}\right\}\right)$$

and

$$p\left(\mu \mid \underline{Y},\sigma_1^2,\sigma_2^2,\sigma_3^2\right) \propto \left[(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^{\frac{-1}{2}} \exp\left(-\frac{1}{2}\frac{IJK(\overline{\overline{\overline{Y}}}_{\ldots} - \mu)^2}{(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)}\right)\right]$$

i.e $\mu \mid \underline{Y} \sim N\left(\overline{\overline{\overline{Y}}}_{\ldots}, \dfrac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{IJK}\right)$

where

$$v_1 m_1 = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk} - \overline{Y}_{ij.})^2,\ v_2 m_2 = K\sum_{i=1}^{I}\sum_{j=1}^{J}(\overline{Y}_{ij.} - \overline{\overline{Y}}_{i..})^2,\ v_3 m_3 = JK\sum_{i=1}^{I}(\overline{\overline{Y}}_{i..} - \overline{\overline{\overline{Y}}}_{\ldots})^2$$

$$\underline{Y} = \left[Y_{111,}\ Y_{112,} \cdots\cdots Y_{IJK}\right]'$$

$v_1 = IJ(K-1), \quad v_2 = I(J-1), \quad v_3 = I-1.$

$IG(x \mid \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{-\alpha-1} \exp(-\beta/x)$ i.e. the inverted gamma density with positive parameters α and β.

and $\sigma_{12}{}^2 = \sigma_1{}^2 + K\sigma_2{}^2, \quad \sigma_{123}{}^2 = \sigma_1{}^2 + K\sigma_2{}^2 + JK\sigma_3{}^2$

<u>Proof</u>

The proof is given in the Appendix.

The posterior distribution for $\sigma_1{}^2, \sigma_{12}{}^2$ and $\sigma_{123}{}^2$ would be independent, each proportional to an inverse gamma distribution, if the restriction $\sigma_{123}{}^2 > \sigma_{12}{}^2 > \sigma_1{}^2$ did not apply. The joint posterior distribution for $\sigma_1{}^2, \sigma_{12}{}^2$ and $\sigma_{123}{}^2$ would be the product of the three distributions.

$$p\left(\sigma_1{}^2, \sigma_{12}{}^2, \sigma_{123}{}^2 \mid \underline{Y}\right) = IG(\sigma_1{}^2 \mid \frac{v_1}{2}; v_1 m_1) \times IG(\sigma_1{}^2 + K\sigma_2{}^2 \mid \frac{v_2}{2}; v_2 m_2) \times IG(\sigma_1{}^2 + K\sigma_2{}^2 + JK\sigma_3{}^2 \mid \frac{v_3}{2}; v_3 m_3)$$

However the restrictions do apply. Nevertheless, using a two-step rejection sampling procedure (as will be discussed), it is straight forward to generate samples from the joint distribution.

Theorem 4.2

I. $E\left(m_1\right) = \sigma_1^2$ **II.** $E\left(m_2\right) = \sigma_{12}{}^2 = (\sigma_1^2 + K\sigma_2{}^2)$ **III.** $E\left(m_3\right) = \sigma_{123}{}^2 = (\sigma_1^2 + K\sigma_2{}^2 + JK\sigma_3{}^2).$

<u>Proof</u>

The proofs are very simple and similar and only III is given in the Appendix.

The following theorem gives the posterior distribution of $\mu_i = \mu + r_i \quad (i = 1, \ldots, I)$ given $\underline{Y}$ and the variance components σ_1^2, σ_2^2 and σ_3^2.

Theorem 6.4.3

μ_i given σ_1^2, σ_2^2 and σ_3^2 is normally distributed with mean

$$E(\mu_i \mid \sigma_1^2, \sigma_2^2, \sigma_3^{\,2}, \underline{Y}) = \frac{JK\sigma_3^{\,2}\overline{\overline{Y_{i..}}}}{(\sigma_1^{\,2} + K\sigma_2^{\,2}) + JK\sigma_3^{\,2}} + \frac{(\sigma_1^{\,2} + K\sigma_2^{\,2})}{(\sigma_1^{\,2} + K\sigma_2^{\,2}) + JK\sigma_3^{\,2}}\overline{\overline{\overline{Y}}}_{...}$$

and variance

$$\mathrm{Var}(\mu_i \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2) = \frac{(\sigma_1^{\,2} + K\sigma_2^{\,2})}{(\sigma_1^{\,2} + K\sigma_2^{\,2} + JK\sigma^2{}_3)}\left\{\frac{IJK\sigma_3^2 + \sigma_1^{\,2} + K\sigma_2^{\,2}}{IJK}\right\}$$

<u>Proof</u>

The proof is given in the Appendix (See van der Merwe and Hugo (2007) for a similar proof).

5. POSTERIOR DISTRIBUTION OF THE LOWER PROCESS PERFORMANCE INDEX $_3\mathbf{P}^1_{\mathbf{pl}}$ WITH THREE VARIANCE COMPONENTS

Theorem 5.1

The posterior distribution of $_3P_{pl}^1$ given the variance components is

$$_3P_{pl}^1 \mid \underline{Y}, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^2 \sim N\left(\frac{\overline{\overline{\overline{Y}}}_{...} - l_0}{3\left(\frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{JK}\right)^{\frac{1}{2}}}, \frac{1}{9I}\right) \text{ for all } i, j, k \qquad (5.1)$$

<u>Proof</u>

The proof is given in the Appendix.

The unconditional posterior distribution of $_3P_{pl}^1$ can be obtained by using Monte Carlo simulation.

6. THE PROBABILITY MATCHING PRIOR FOR THE LOWER

PROCESS CAPABILITY INDEX ${}_3P^1_{pl}$

Theorem 6.1

The probability-matching prior for the ${}_3P^1_{pl}$ Index for the balanced random effects model defined in equation 3.1 is:

$$\pi(\underline{\theta}) = \pi(\mu, \sigma_1^2, \sigma_2^2, \sigma_3^2) \propto \sigma_1^{-2} \left(\sigma_1^2 + K\sigma_2^2\right)^{-1} \left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{-\frac{1}{2}} \left(1 + \frac{(\mu - l_0)^2 (JK)}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}\right)^{\frac{1}{2}}$$

Proof

The proof is given in the Appendix.

7. MONTE CARLO SIMULATION PROCEDURE FOR ESTIMATING THE POSTERIOR DISTRIBUTION OF ${}_3P^1_{pl}$

Simulation of the posterior of σ_1^2, σ_2^2 and σ_3^2 can be achieved through the following standard simulation routines. By using the Matlab package, simulation of σ_1^2, σ_2^2 and σ_3^2 can be obtained in the following way:

i. Draw τ from a χ_{v_1} distribution

ii. Put $\frac{1}{\tau} = \frac{\sigma_1^2}{v_1 m_1}$

iii. $\sigma_1^{2*} = \frac{v_1 m_1}{\tau}$ *where the * indicates a simulated value.*

iv. Draw u from a χ_{v_2} distribution

v. Put $\frac{1}{u} = \frac{\sigma_{12}^2}{v_2 m_2}$ *where* $\sigma_{12}^2 = \sigma_1^2 + J\sigma_2^2$

vi. $\sigma_{12}{}^{2*} = \frac{v_2 m_2}{u}$

vii. If $\sigma_{12}{}^{2*} > \sigma_1{}^{2*}$ $(\sigma_1{}^{2*}$ *was simulated in step* $(iii))$ Calculate

Calculate

$$\sigma_2{}^{2*} = \frac{1}{J}(\sigma_{12}{}^{2*} - \sigma_1{}^{2*}) \quad \textit{from the expression} \quad \sigma_{12}{}^{2*} = \sigma_1{}^{2*} + J\sigma_2{}^{2*}.$$

If $\sigma_{12}{}^{2*} < \sigma_1{}^{2*}$ ignore the values of $\sigma_1{}^{2*}$ *and* $\sigma_2{}^{2*}$ and start again.

viii. Draw $\aleph$ from a χ_{v_3} distribution

ix. Put $\frac{1}{\aleph} = \frac{\sigma_{123}{}^2}{v_3 m_3}$ *where* $\sigma_{123}{}^2 = \sigma_1{}^2 + K\sigma_2{}^2 + JK\sigma_3{}^2$

x. $\sigma_{123}{}^{2*} = \frac{v_3 m_3}{\aleph}$

xi. *If* $\sigma_{123}{}^{2*} > \sigma_{12}{}^{2*} > \sigma_1{}^{2*}$ $(\sigma_1{}^{2*}$ *was simulated in step* (iii) $\sigma_2{}^{2*}$ *was simulated in step* $(vii))$

Calculate

$$\sigma_3{}^{2*} = \frac{\sigma_{123}{}^{2*} - [\sigma_1{}^{2*} + K\sigma_2{}^{2*}]}{JK} \quad \textit{from the expression} \quad \sigma_{123}{}^{2*} = \sigma_1{}^{2*} + K\sigma_2{}^{2*} + JK\sigma_3{}^{2*}.$$

If $\sigma_{123}{}^{2*} < \sigma_{12}{}^{2*}$ ignore the values of $\sigma_1{}^{2*}, \sigma_2{}^{2*}$ *and* $\sigma_3{}^{2*}$ and start again.

By making use of the fact that $\mu \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N\left\{ \overline{\overline{\overline{Y}}}_{\ldots}, \frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{IJK} \right\}$ and from the definition of the Performance index,

$${}_3P_{pl}^{1\,*} = \frac{\mu^* - l_0}{3\left(\frac{\sigma_1^{2*} + K\sigma_2^{2*} + JK\sigma_3^{2*}}{JK} \right)^{\frac{1}{2}}}$$

it follows that ${}_3P_{pl}^1$ can be simulated.

Given $\sigma_1^2, \sigma_2^2, \sigma_3^2$, the conditional posterior density function $p({}_3P_{pl}^1 \mid \underline{Y}, \sigma_1{}^2, \sigma_2{}^2, \sigma_3{}^2)$ is calculated which is defined in (5.1). Repeat steps (i-xi) $\tilde{\ell}$ times. For our example $\tilde{\ell}$ was taken as 10000. Using a Rao–Blackwell argument (Gelfand and Smith,1991), a density estimate of the unconditional posterior distribution of ${}_3P_{pl}^1 = \dfrac{\overline{\overline{\overline{Y}}}_{...} - l_0}{3\left(\dfrac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{JK}\right)^{\frac{1}{2}}}$ can be obtained by averaging $p({}_3P_{pl}^1 \mid \underline{Y}, \sigma_1{}^2, \sigma_2{}^2, \sigma_3{}^2)$ over the $\tilde{\ell}$ repetitions.

Simulation results for the variance component process performance index ${}_3P_{pl}^1$ as discussed in this article will now be compared with the following indices
${}_3P_{pl}{}^{11}, {}_3P_{pl}{}^{11}$ and ${}_3P_{pl}{}^{1V}$.

By making use of the fact that
$\mu_i \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N\left\{E(\mu_i \mid \sigma_1^2, \sigma_2^2, \sigma_3{}^2, \underline{Y}), \text{Var}(\mu_i \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2)\right\}$
where

$$E(\mu_i \mid \sigma_1^2, \sigma_2^2, \sigma_3{}^2, \underline{Y}) = \frac{JK\sigma_3{}^2 \overline{\overline{Y_{i..}}}}{(\sigma_1{}^2 + K\sigma_2{}^2) + JK\sigma_3{}^2} + \frac{(\sigma_1{}^2 + K\sigma_2{}^2)}{(\sigma_1{}^2 + K\sigma_2{}^2) + JK\sigma_3{}^2} \overline{\overline{\overline{Y}}}_{...} \quad \text{and}$$

$$\text{Var}(\mu_i \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2) = \frac{(\sigma_1{}^2 + K\sigma_2{}^2)}{(\sigma_1{}^2 + K\sigma_2{}^2 + JK\sigma^2{}_3)} \left\{\frac{IJK\sigma_3^2 + \sigma_1{}^2 + K\sigma_2{}^2}{IJK}\right\}$$

and from the definition of the index, ${}_3P_{pl}^V$ can also be simulated.

$${}_3P_{pl}^{V*} = \left(\frac{\mu_i^* - l_0}{3\left(\dfrac{\sigma_1{}^{2*} + K\sigma_2{}^{2*}}{JK}\right)^{\frac{1}{2}}}\right).$$

Simulation results using the probability matching prior will be considered next.

8. **THE WEIGHTED MONTE CARLO METHOD -SAMPLING-IMPORTANCE RE-SAMPLING**

This section describes how to apply a weighted Monte Carlo (WMC) method to simulate ${}_3P_{pl}^1$ using the probability matching prior. This method is especially suitable for computing credibility intervals.

Let

$$\pi(\underline{\theta}) \propto \sigma_1^{-2}\left(\sigma_1^2 + K\sigma_2^2\right)^{-1}\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{-\frac{1}{2}}\left(1+\frac{(\mu-l_0)^2(JK)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right)^{\frac{1}{2}} \qquad (8.1)$$

and

$$q(\underline{\theta}) \propto \sigma_1^{-2}(\sigma_1^2 + K\sigma_2^2)^{-1}(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^{-1} \qquad (8.2)$$

According to Smith and Gelfand (1992), Guttman and Menzefricke (2003) and Skare *et al.* (2003), the $\tilde{\ell}$ independent draws of $\underline{\theta}^{*(\ell)} = (\mu^{*(\ell)}, \sigma_1^{2*(\ell)}, \sigma_2^{2*(\ell)}, \sigma_3^{2*(\ell)})$, as discussed in section 7, for $\ell = 1$ to $\tilde{\ell}$; is a weighted sample from the posterior distribution based on π, where the weights are

$$w_\ell = \frac{\pi(\underline{\theta}^{*(\ell)}) / q(\underline{\theta}^{*(\ell)})}{\sum_{\ell=1}^{\tilde{\ell}} \pi(\underline{\theta}^{*(\ell)}) / q(\underline{\theta}^{*(\ell)})}$$

$\pi(\underline{\theta}^{*(\ell)})$ and $q(\underline{\theta}^{*(\ell)})$ denote the realistic prior density and implied prior density defined in equations (8.1) and (8.2). For the algorithm to be efficient, it is important that q is a good approximation to π. This means that q should not have too light tails when compared to π. For further details see Skare *et al* (2003) and Li (2007).

To simulate using the results from the probability matching prior we associate with each lower performance index value ${}_3P_{pl}^{1*(\ell)}$

$$w_\ell = \frac{\sigma_1^{-2*(\ell)}\left(\sigma_1^{2*(\ell)}+K\sigma_2^{2*(\ell)}\right)^{-1}\left(\sigma_1^{2*(\ell)}+K\sigma_2^{2*(\ell)}+JK\sigma_3^{2*(\ell)}\right)^{-\frac{1}{2}}\left(1+\frac{(\mu^{*(\ell)}-l_0)^2(JK)}{2\left(\sigma_1^{2*(\ell)}+K\sigma_2^{2*(\ell)}+JK\sigma_3^{2*(\ell)}\right)}\right)^{\frac{1}{2}}}{\sum_{\ell=1}^{\tilde{\ell}}\sigma_1^{-2*(\ell)}\left(\sigma_1^{2*(\ell)}+K\sigma_2^{2*(\ell)}\right)^{-1}\left(\sigma_1^{2*(\ell)}+K\sigma_2^{2*(\ell)}+JK\sigma_3^{2*(\ell)}\right)^{-\frac{1}{2}}\left(1+\frac{(\mu^{*(\ell)}-l_0)^2(JK)}{2\left(\sigma_1^{2*(\ell)}+K\sigma_2^{2*(\ell)}+JK\sigma_3^{2*(\ell)}\right)}\right)^{\frac{1}{2}}}$$

a. Sort the ${}_3P_{pl}^{1*(\ell)}$ values calculated in ascending order so that

$${}_3P_{pl}^{1\,*(1)} \le {}_3P_{pl}^{1\,*(2)} \le \cdots \le {}_3P_{pl}^{1\,*(10000)}$$

b. Compute the weighted function w_ℓ associated with the ℓth ordered $P_{pl}^{1*(\ell)}$.

c. Add up the weights from left to right (from the first on) till you get $\sum_{\ell=1}^{k_1} w_\ell = 0.025$. Write down the corresponding ordered value ${}_3P_{pl}^{1*(k_1)}$ and denote it as $P_{pl\;(0.025)}^{1*}$. Add up the weights from right to left (from the last back) till you get $\sum_{\ell=k_2}^{\tilde{\ell}} w_\ell = 0.025$. Write down the corresponding ordered value ${}_3P_{pl}^{1\,*(k_2)}$ and denote it as ${}_3P_{pl\,(0.975)}^{1*}$. The 95% interval is ${}_3P_{pl\,(0.025)}^{1*} - {}_3P_{pl\,(0.975)}^{1*}$.

9. APPLICATION

In a process to manufacture chronic medication, various properties of the manufactured tablet have to be monitored. Monthly samples of J=8 packages of the tablet are sampled and various physical properties of the tablet are replicated in the laboratory by analysing K=5 tablets per package. The data in Table 1 represents the amount of drug in a tablet (the percentage of the drug per tablet). Packages with tablets sampled for the first I=15 months starting January of a particular year are selected as review data to determine whether the patient gets on average the required dosage of the drug from the batches in a specified time, given that each patient must get an average dosage of at least $l_0 = 20$.

Table1: Drug data arising from multiple sources of variability

Day	Package	Average Package Dosage	Day	Package	Average Package Dosage	Day	Package	Average Package Dosage
1	1	23.3000	1	6	24.7900	1	11	24.4700
1	2	23.0600	2	6	24.6500	2	11	23.4200
1	3	23.4800	3	6	24.7500	3	11	23.6100
1	4	22.2200	4	6	24.9400	4	11	22.9100
1	5	22.1600	5	6	24.7700	5	11	22.8600
1	6	22.9400	6	6	25.1700	6	11	23.5700
1	7	23.6900	7	6	24.6700	7	11	23.6100
1	8	22.0200	8	6	24.5100	8	11	22.3700
2	1	24.9100	1	7	24.5200	1	12	22.2200
2	2	24.6800	2	7	23.4400	2	12	23.5900
2	3	25.0900	3	7	23.8800	3	12	23.6800
2	4	25.3900	4	7	23.1300	4	12	23.2300
2	5	24.5600	5	7	23.6400	5	12	23.9300
2	6	25.3800	6	7	23.5800	6	12	23.9800
2	7	24.8200	7	7	23.5000	7	12	23.6900
2	8	24.9000	8	7	23.7100	8	12	23.4600
3	1	25.2900	1	8	25.0900	1	13	24.5800
3	2	24.1900	2	8	23.7100	2	13	25.2700
3	3	24.9000	3	8	23.6800	3	13	24.5200
3	4	24.7000	4	8	23.6700	4	13	25.6200

3	5	24.6900	5	8	23.6700	5	13	24.7300
3	6	24.6500	6	8	23.9800	6	13	24.8000
3	7	24.8500	7	8	23.9400	7	13	25.0000
3	8	24.5000	8	8	24.1300	8	13	24.5300
4	1	23.8600	1	9	22.1900	1	14	25.3100
4	2	24.1200	2	9	22.8400	2	14	24.8400
4	3	23.2000	3	9	22.9500	3	14	25.2900
4	4	23.3600	4	9	23.2500	4	14	25.3800
4	5	23.6400	5	9	24.6000	5	14	25.4000
4	6	22.8500	6	9	22.6500	6	14	24.6000
4	7	22.7200	7	9	23.7700	7	14	25.1400
4	8	23.3300	8	9	23.3700	8	14	23.9400
5	1	23.4800	1	10	23.9900	1	15	25.4800
5	2	23.4300	2	10	23.5300	2	15	24.6500
5	3	23.0700	3	10	22.3700	3	15	25.0000
5	4	22.6900	4	10	22.9300	4	15	25.7100
5	5	23.6000	5	10	22.3100	5	15	25.2800
5	6	23.4900	6	10	22.8600	6	15	24.3800
5	7	23.4600	7	10	22.0000	7	15	24.3500
5	8	22.7400	8	10	22.8100	8	15	25.4300

In addition, $v_1 m_1 = 390.6720$. The above data and limit is selected solely for illustrative purposes. In practice, fixed in advance limits are often determined from medical or regulatory considerations.

As introduced above, the variation observed could possibly be explained by several components such as a “between” months (month-to-month) component, a “within” month (package-to-package) component and a residual component.

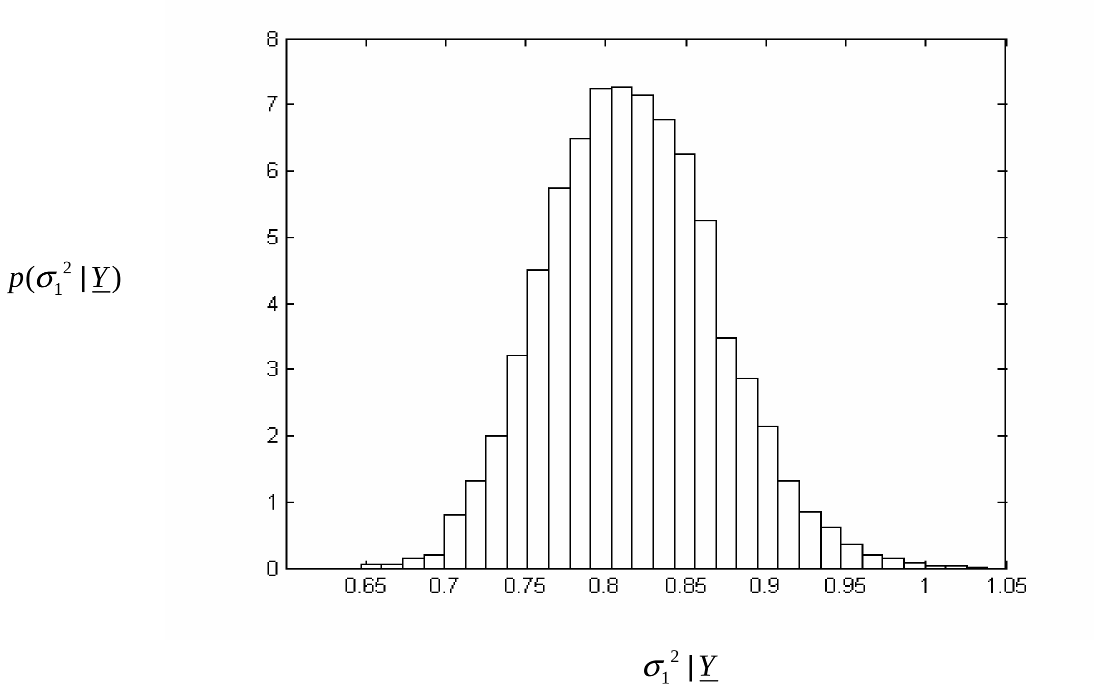
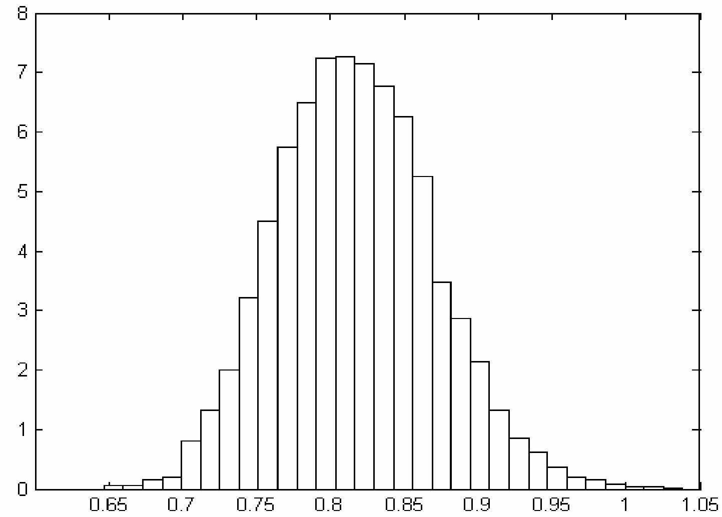


Figure 1: Histogram of simulated variance component

A plot of the posterior distribution of $\sigma_1^2 \mid \underline{Y}$ is symmetrical or fairly symmetrical. The reason for this is the large number of degrees of freedom $v_1 = IJ(K-1) = 480$ associated with the residual variance.



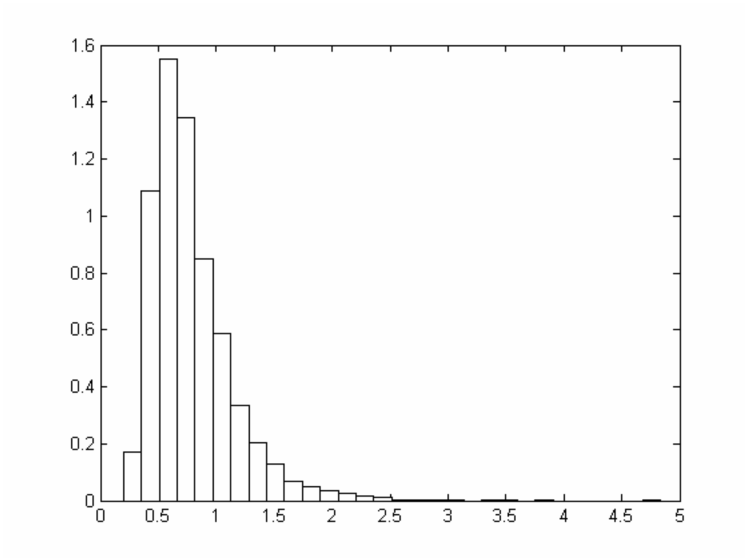
$p(\sigma_2^{\ 2} \mid \underline{Y})$

$\sigma_2^{\ 2} \mid \underline{Y}$

Figure 2: Histogram of simulated variance component

A plot the posterior distribution of $\sigma_2^{\ 2} \mid \underline{Y}$ is also fairly symmetrical. The reason for this is again the large number of degrees of freedom $v_2 = I(J-1) = 105$ associated with the residual variance.

$p(\sigma_3^2 \mid \underline{Y})$



$\sigma_3^2 \mid \underline{Y}$

Figure 3: Histogram of simulated variance component

The posterior distribution of $\sigma_3^2 \mid \underline{Y}$ on the other hand is quite skewed. The between months variation is much larger than the within months variation.

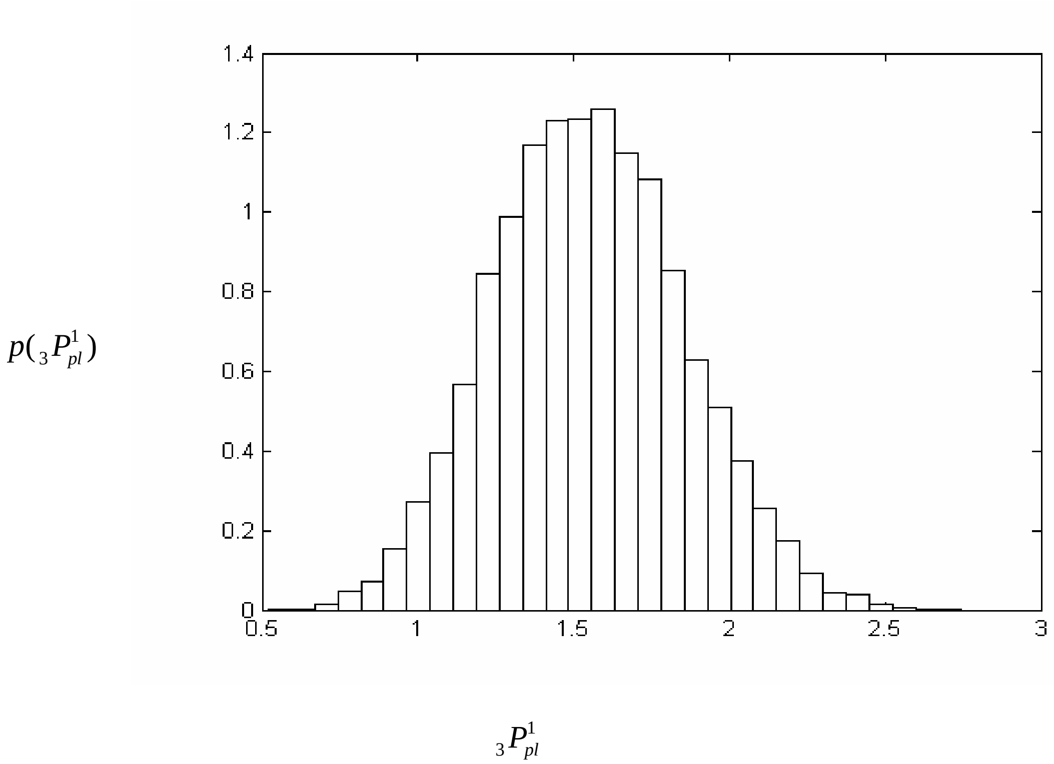


Figure 4: Histogram of simulated index

The mean of the index ${}_3P_{pl}^{1}$ is 1.5499 showing that the process is capable.

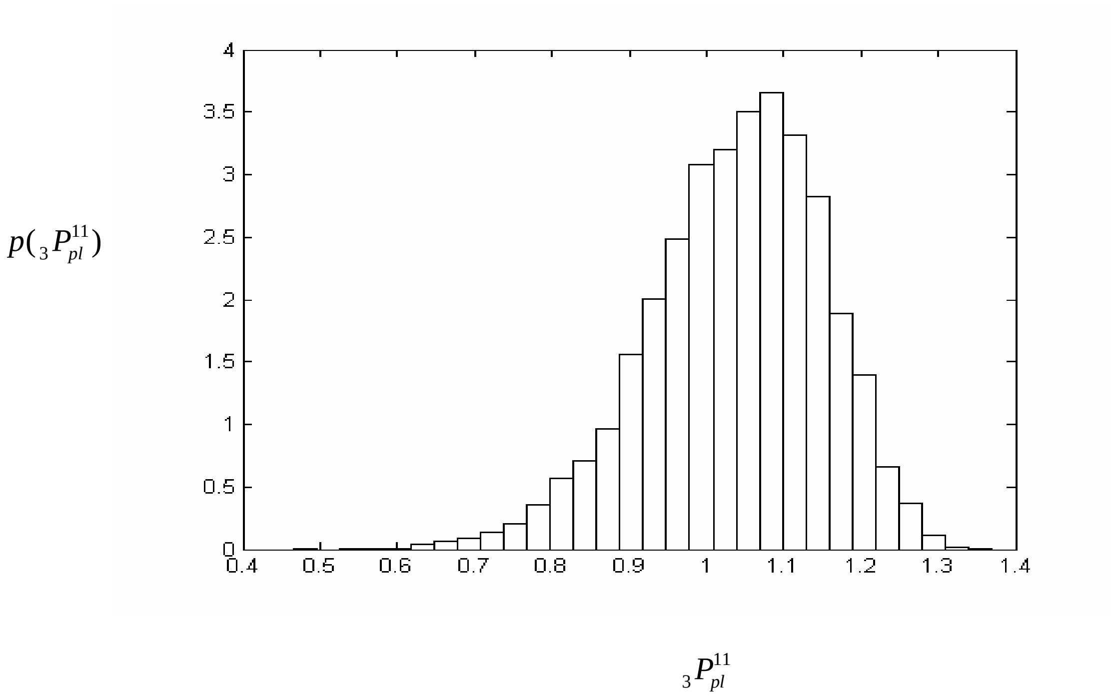


Figure 5: Histogram of simulated index

The mean of the index ${}_3P_{pl}^{11}$ is 1.0385 showing that the process is capable.

The results of the indices are summarised in the table below.

Table 2: Posterior mean, variance and 95% interval for the indices using the non-informative Jeffery's prior

Index	Mean	Variance	95% Interval
${}_3P_{pl}^{1}$	1.5499	0.0925	(0.9861;2.1634)
${}_3P_{pl}^{11}$	1.0385	0.0126	(0.7936;1.2307)
${}_3P_{pl}^{111}$	1.3642	0.0471	(0.9304;1.7672)
${}_3P_{pl}^{1V}$	6.0029	1.3873	(3.8192;8.3787)
${}_3P_{pl}^{V}$	5.8081	0.1059	(5.1653;6.4356)

The corresponding 95% interval in the case of the probability-matching prior for ${}_3P_{pl}{}^{1}$ is (0.9800;2.1654).

Some features of these results worth noting are:

- The mean of the indices are greater than1 suggesting that the process is capable.
- The mean of the Indices ${}_3P_{pl}^{1V}$ and ${}_3P_{pl}^{V}$ are much greater than 1 suggesting that the process is very capable when a patient takes the tablets for longer periods. In fact the data suggest that the process is super.
- The variance of the index ${}_3P_{pl}^{11}$ is small when compared to the variances of the other Indices.

It has been suggested that some advanced patients may require a slightly higher dosage of the drug. The question therefore is whether the process is capable of producing to specification under these same machine settings and for $l_0 = 21,\ 22$ and 23.

$l_0 = 21$.

Table 3: Posterior mean, variance and 95% interval for the Indices using the non-informative Jeffery's prior

Index	Mean	Variance	95% Interval
${}_3P_{pl}^{1}$	1.1584	0.0548	(0.7208;1.6302)
${}_3P_{pl}^{11}$	0.7762	0.0086	(0.5744;0.9359)
${}_3P_{pl}^{111}$	1.0196	0.0288	(0.6777;1.3368)
${}_3P_{pl}^{1V}$	4.4864	0.8218	(2.7918;6.3136)
${}_3P_{pl}^{V}$	3.8008	0.0583	(3.3201;4.2691)

The corresponding 95% interval in the case of the probability-matching prior for ${}_3P_{pl}{}^{1}$ is (0.7199;1.6318).

Some features of these results worth noting are:

- The mean of the indices ${}_3P_{pl}{}^{11}$ is less than 1 suggesting that the process is not capable of producing the individual tablets to specification but the other indices are greater than 1 suggesting that the process is capable once the tablets are analysed at least as a batch.
- The variance of the indices ${}_3P_{pl}{}^{11}$ and ${}_3P_{pl}{}^{111}$ are smaller when compared to the variances of the other Indices.

$l_0 = 22.$

Table 4: Posterior mean, variance and 95% interval for the Indices using the non-informative Jeffery's prior

Index	Mean	Variance	95% Interval
${}_3P_{pl}^{1}$	0.7669	0.0280	(0.4470;1.004)
${}_3P_{pl}^{11}$	0.5138	0.0057	(0.3533;0.6469)
${}_3P_{pl}^{111}$	0.6750	0.0158	(0.4196;0.9129)
${}_3P_{pl}^{1V}$	2.9700	0.4203	(1.7314;4.2619)
${}_3P_{pl}^{V}$	1.7935	0.0315	(1.4390;2.1314)

The corresponding 95% interval in the case of the probability-matching prior for ${}_3P_{pl}{}^{1}$ is (0.4465;1.1015).

Some features of these results worth noting are:

- The mean of the indices ${}_3P_{pl}{}^{1}$, ${}_3P_{pl}{}^{11}$ and ${}_3P_{pl}{}^{111}$ are less than 1 suggesting that the process is not capable but the other indices are greater than 1 suggesting that the process is capable. The process is capable once the tablets are taken over longer periods.
- The variance of the indices ${}_3P_{pl}{}^{11}$ and ${}_3P_{pl}{}^{111}$ are smaller when compared to the variances of the other Indices.

$l_0 = 23.$

Table 5: Posterior mean, variance and 95% interval for the indices using the non-informative Jeffery's prior

Index	Mean	Variance	95% Interval
${}_3P_{pl}^{1}$	0.3753	0.0122	(0.1636;0.5922)
${}_3P_{pl}^{11}$	0.2515	0.0039	(0.1243;0.3668)
${}_3P_{pl}^{111}$	0.3304	0.0081	(0.1525;0.5000)
${}_3P_{pl}^{1V}$	1.4536	0.1827	(0.6338;2.2937)
${}_3P_{pl}^{V}$	-0.2138	0.0254	(-0.5347; 0.0851)

The corresponding 95% interval in the case of the probability-matching prior for ${}_3P_{pl}{}^{1}$ is (0.1631; 0.5931).

Some features of these results worth noting are:

- The mean of all the indices except ${}_3P_{pl}{}^{1V}$ are less than 1 suggesting that the process is not capable. The process is capable once the tablets are taken over longer periods of I months
- The variance of the indices ${}_3P_{pl}{}^{11}$ and ${}_3P_{pl}{}^{111}$ are smaller when compared to the variances of the other indices.
- The index ${}_3P_{pl}{}^{V}$ which was been giving the second highest figure all along now suddenly becomes the lowest and even negative. This is because once we know the month, the third variance component falls away and the index is computed using a smaller variance and becomes very sensitive to departures from the mean.

Appendix A6

Proof of theorem 6.3.1

III.

$$\textit{If we define} \quad \overline{\overline{Y}}_{i..} = \frac{\sum_{j=1}^{J}\sum_{k=1}^{K} Y_{ijk}}{KJ}$$

$$E[\overline{\overline{Y}}_{i..} \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2}] = E[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K} Y_{ijk}}{KJ} \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2}]$$

$$= E[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K} (\mu + r_i + c_{ij} + \varepsilon_{ijk})}{KJ} \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2}]$$

$$= E[\frac{KJ\mu}{KJ} \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2}] + E[\frac{KJr_i}{KJ} \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2}] +$$

$$E[\frac{K\sum_{j=1}^{J} (c_{ij} \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2})}{KJ}] + E[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K} (\varepsilon_{ijk} \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2})}{KJ}]$$

$$= E[\mu \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2}] + E[r_i \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2}] + [\frac{\sum_{j=1}^{J} E(c_{ij} \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2})}{J}]$$

$$+[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K} E(\varepsilon_{ijk} \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2})}{KJ}]$$

$$= \mu + 0 + 0 + 0 = \mu$$

$$V[\overline{\overline{Y}}_{i..} \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2}] = V[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K} Y_{ijk}}{JK} \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2}]$$

$$= V[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K} (\mu + r_i + c_{ij} + \varepsilon_{ijk})}{JK} \mid \mu, \sigma_1^{\,2}, \sigma_2^{\,2}, \sigma_3^{\,2}]$$

$$=V[\frac{JK\mu}{JK}\,|\,\mu,\sigma_1{}^2,\sigma_2{}^2,\sigma_3{}^2]+V[\frac{JKr_i}{JK}\,|\,\mu,\sigma_1{}^2,\sigma_2{}^2,\sigma_3{}^2]+$$

$$V[\frac{K\sum_{j=1}^{J}(c_{ij}\,|\,\mu,\sigma_1{}^2,\sigma_2{}^2,\sigma_3{}^2)}{JK}]+V[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}(\varepsilon_{ijk}\,|\,\mu,\sigma_1{}^2,\sigma_2{}^2,\sigma_3{}^2)}{JK}]$$

$$=V[\mu\,|\,\mu,\sigma_1{}^2,\sigma_2{}^2,\sigma_3{}^2]+V[r_i\,|\,\mu,\sigma_1{}^2,\sigma_2{}^2,\sigma_3{}^2]+$$

$$[\frac{\sum_{j=1}^{J}V(c_{ij}\,|\,\mu,\sigma_1{}^2,\sigma_2{}^2,\sigma_3{}^2)}{J^2}]+[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}V(\varepsilon_{ijk}\,|\,\mu,\sigma_1{}^2,\sigma_2{}^2,\sigma_3{}^2)}{J^2K^2}]$$

$$=0+\sigma_3{}^2+\frac{J\sigma_2{}^2}{J^2}+\frac{JK\sigma_1{}^2}{J^2K^2}=\sigma_3{}^2+\frac{\sigma_2{}^2}{J}+\frac{\sigma_1{}^2}{JK}=\frac{KJ\sigma_3{}^2+K\sigma_2{}^2+\sigma_1{}^2}{JK}$$

V.

$$Define \quad \overline{\overline{Y_{i..}}}=\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}Y_{ijk}}{KJ}$$

$$E[\overline{\overline{Y_{i..}}}\,|\,\theta,r_i,\sigma_1{}^2,\sigma_2{}^2]=E[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}Y_{ijk}}{KJ}\,|\,\theta,r_i,\sigma_1{}^2,\sigma_2{}^2]$$

$$=E[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}(\theta+r_i+c_{ij}+\varepsilon_{ijk})}{KJ}\,|\,\theta,r_i,\sigma_1{}^2,\sigma_2{}^2]$$

$$= E[\frac{KJ\theta}{KJ} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2] + E[\frac{KJr_i}{KJ} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2] +$$

$$E[\frac{K\sum_{j=1}^{J}(c_{ij} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2)}{KJ}] + E[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}(\varepsilon_{ijk} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2)}{KJ}]$$

$$= E[\theta | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2] + E[r_i | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2] + [\frac{\sum_{j=1}^{J}E(c_{ij} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2)}{J}]$$

$$+[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}E(\varepsilon_{ijk} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2)}{KJ}]$$

$$= \theta + r_i + 0 + 0$$

$$= \theta + r_i$$

$$V[\overline{\overline{Y}}_{i..} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2] = V[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}Y_{ijk}}{JK} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2]$$

$$= V[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}(\theta + r_i + c_{ij} + \varepsilon_{ijk})}{JK} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2]$$

$$= V[\frac{JK\theta}{JK} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2] + V[\frac{JKr_i}{JK} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2] +$$

$$V[\frac{K\sum_{j=1}^{J}(c_{ij} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2)}{JK}] + V[\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}(\varepsilon_{ijk} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2)}{JK}]$$

$$= V[\theta | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2] + V[r_i | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2] +$$

$$[\frac{\sum_{j=1}^{J}V(c_{ij} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2)}{J^2}] + [\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}V(\varepsilon_{ijk} | \theta, r_i, \sigma_1{}^2, \sigma_2{}^2)}{J^2K^2}]$$

$$= 0 + 0 + \frac{J\sigma_2{}^2}{J^2} + \frac{JK\sigma_1{}^2}{J^2K^2}$$

$$=\frac{\sigma_2{}^2}{J}+\frac{\sigma_1{}^2}{JK}$$

$$=\frac{\sigma_2{}^2}{J}+\frac{\sigma_1{}^2}{JK}$$

$$=\frac{K\sigma_2{}^2+\sigma_1{}^2}{JK}$$

Proof of theorem 6.4.1

The non-informative joint prior:

$$p(\mu,\sigma_1{}^2,\sigma_{12}{}^2,\sigma_{123}{}^2)\propto p(\mu)p(\sigma_1{}^2,\sigma_{12}{}^2,\sigma_{123})$$

$$=c\times p(\sigma_1{}^2,\sigma_{12}{}^2,\sigma_{123})$$

$$\propto \sigma_1^{-2}(\sigma_1^2+K\sigma_2^2)^{-1}(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^{-1},$$

The joint posterior distribution of μ and $\sigma_1{}^2,\sigma_2{}^2$ and σ_3^2 can be worked out.

The posterior is computed as follows:

Let $\underline{Y}=[Y_{111,}Y_{112,}Y_{113,}\cdots Y_{IJK}]$

Posterior $\propto$ *likelihood x prior*

$$p\left(\mu,\sigma_1{}^2,\sigma_{12}{}^2,\sigma_{123}\,|\,\underline{Y}\right)\propto\prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{k=1}^{K}f(Y_{ijk}\,|\,\mu,\sigma_1{}^2,\sigma_2{}^2,\sigma_3{}^2)\,p\left(\mu,\sigma_1{}^2,\sigma_{12}{}^2,\sigma_{123}\right)$$

$$=\left[(\sigma_1{}^2+\sigma_2{}^2+\sigma_3{}^2)^{\frac{-IJK}{2}}\exp(-\frac{1}{2}\{\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk}-\mu)^2}{(\sigma_1{}^2+\sigma_2{}^2+\sigma_3{}^2)}\}\right]\frac{1}{\sigma_1^2(\sigma_1^2+K\sigma_2^2)(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)}$$

$$\propto\sigma_1{}^{-\frac{1}{2}v_1}(\sigma_1{}^2+K\sigma_2{}^2)^{-\frac{1}{2}v_2}(\sigma_1{}^2+K\sigma_2{}^2+JK\sigma_3{}^2)^{-\frac{1}{2}(v_3+1)}\times$$

$$\left[\exp(-\frac{1}{2}\{\frac{v_1m_1}{\sigma_1{}^2}+\frac{v_2m_2}{(\sigma_1{}^2+K\sigma_2{}^2)}+\frac{v_3m_3}{(\sigma_1{}^2+K\sigma_2{}^2+JK\sigma_3{}^2)}+\frac{IJK(\overline{\overline{Y}}_{\cdots}-\mu)^2}{(\sigma_1{}^2+K\sigma_2{}^2+JK\sigma_3{}^2)}\}\right]\times$$

$$\frac{1}{\sigma_1^2(\sigma_1^2+K\sigma_2^2)(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)}$$

$$\propto (\sigma_1^2)^{-\frac{1}{2}(\nu_1+2)}(\sigma_1^2+K\sigma_2^2)^{-\frac{1}{2}(\nu_2+2)}(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^{-\frac{1}{2}(\nu_3+3)}\times$$

$$\exp\left\{-\frac{1}{2}\left[\frac{IJK(\overline{\overline{\overline{Y}}}_{...}-\theta)^2}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}+\frac{\nu_3 m_3}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}+\frac{\nu_2 m_2}{\sigma_1^2+K\sigma_2^2}+\frac{\nu_1 m_1}{\sigma_1^2}\right]\right\}.$$

This is the joint posterior of all the parameters.

$$p\left(\mu,\sigma_1{}^2,\sigma_{12}{}^2,\sigma_{123}{}^2 \mid \underline{Y}\right)\propto\left[(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^{\frac{-1}{2}}\exp(-\frac{1}{2}\frac{IJK(\overline{\overline{\overline{Y}}}_{...}-\mu)^2}{(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)})\right]$$

$$\left((\sigma_1^2)^{-\frac{1}{2}(\nu_1+2)}(\sigma_1^2+K\sigma_2^2)^{-\frac{1}{2}(\nu_2+2)}(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^{-\frac{1}{2}(\nu_3+2)}\times \exp(-\frac{1}{2}\{\frac{\nu_3 m_3}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}+\frac{\nu_2 m_2}{\sigma_1^2+K\sigma_2^2}+\frac{\nu_1 m_1}{\sigma_1^2}\})\right)$$

The function inside the curled brackets on the right hand side is therefore proportional to the marginal joint distribution of the variance components $\sigma_1{}^2,\sigma_2{}^2,\sigma_3{}^2$.

The joint posterior distribution of the variance components is given by

$$p(\sigma_1{}^2,\sigma_{12}{}^2,\sigma_{123}{}^2 \mid \underline{Y})\propto \quad (\sigma_1^2)^{-\frac{1}{2}(\nu_1+2)}(\sigma_1^2+K\sigma_2^2)^{-\frac{1}{2}(\nu_2+2)}(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^{-\frac{1}{2}(\nu_3+2)}\times$$
$$\exp\left\{-\frac{1}{2}\left[\frac{\nu_3 m_3}{\sigma_1^2+K\sigma_2^2+JK\sigma_1^2}+\frac{\nu_2 m_2}{\sigma_1^2+K\sigma_2^2}+\frac{\nu_1 m_1}{\sigma_1^2}\right]\right\}$$

$$p\left(\sigma_1{}^2,\sigma_{12}{}^2,\sigma_{123}{}^2 \mid \underline{Y}\right)=\left(\begin{array}{l}(\sigma_1{}^2)^{\frac{-(\frac{1}{2}\nu_1+2)}{2}}\exp(-\frac{1}{2}\{\frac{\nu_1 m_1}{\sigma_1{}^2}\})\times(\sigma_1{}^2+K\sigma_2{}^2)^{\frac{-\frac{1}{2}(\nu_2+2)}{2}}\exp(-\frac{1}{2}\{\frac{\nu_2 m_2}{(\sigma_1{}^2+K\sigma_2{}^2)}\}\times \\ (\sigma_1{}^2+K\sigma_2{}^2+JK\sigma_3{}^2)^{\frac{-\frac{1}{2}(\nu_3+2)}{2}}\exp(-\frac{1}{2}\{\frac{\nu_3 m_3}{(\sigma_1{}^2+K\sigma_2{}^2+JK\sigma_3{}^2)}\})\end{array}\right)$$

i.e.

$$p\left(\sigma_1{}^2,\sigma_{12}{}^2,\sigma_{123}{}^2 \mid \underline{Y}\right)=IG(\sigma_1{}^2 \mid \frac{\nu_1}{2};\nu_1 m_1)\times IG(\sigma_1{}^2+K\sigma_2{}^2 \mid \frac{\nu_2}{2};\nu_2 m_2)\times IG(\sigma_1{}^2+K\sigma_2{}^2+JK\sigma_3{}^2 \mid \frac{\nu_3}{2};\nu_3 m_3)$$

where

$$IG(x \mid \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{-\alpha-1} \exp(-\beta/x)$$

i.e. the inverted gamma density with positive parameters α and β.

The function inside the square brackets just above, when regarded as a function of μ, is proportional to the conditional distribution of a normal distribution for which the mean is:

$$E(\mu \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2) = \overline{\overline{\overline{Y}}}_{\ldots}$$

and variance:

$$\text{Var}(\mu \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2) = \frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{IJK}$$

i.e $\mu \mid \underline{Y} \sim N(\overline{\overline{\overline{Y}}}_{\ldots}, \frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{IJK})$

Proof of theorem 6.4.2

III. We show that $E(m_3) = (\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)$

$$v_3 m_3 = JK \sum_{i=1}^{I} (\overline{\overline{Y}}_{i..} - \overline{\overline{\overline{Y}}}_{\ldots})^2$$

$$E(v_3 m_3) = JK \sum_{i=1}^{I} E(\overline{\overline{Y}}_{i..} - \overline{\overline{\overline{Y}}}_{\ldots})^2$$

$$E(v_3 m_3) = JK \sum_{i=1}^{I} E\left((\overline{\overline{Y}}_{i..} - \mu) - (\overline{\overline{\overline{Y}}}_{\ldots} - \mu) \right)^2$$

$$E(v_3 m_3) = JK \sum_{i=1}^{I} E\left((\overline{\overline{Y}}_{i..} - \mu)^2 + (\overline{\overline{\overline{Y}}}_{\ldots} - \mu)^2 - 2(\overline{\overline{Y}}_{i..} - \mu)(\overline{\overline{\overline{Y}}}_{\ldots} - \mu) \right)$$

$$E(v_3 m_3) = JK \sum_{i=1}^{I} \left(E(\overline{\overline{Y}}_{i..} - \mu)^2 + E((\overline{\overline{\overline{Y}}}_{\ldots} - \mu)^2)^2 - 2E(\overline{\overline{Y}}_{i..} - \mu)((\overline{\overline{\overline{Y}}}_{\ldots} - \mu)^2) \right)$$

$$E(v_3 m_3) = JK \sum_{i=1}^{I} \begin{pmatrix} Var(\overline{\overline{Y}}_{i..} \mid \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2) + Var(\overline{\overline{\overline{Y}}}_{\ldots} \mid \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2) \\ -2Cov(\overline{\overline{Y}}_{i..} \mid \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2, \overline{\overline{\overline{Y}}}_{\ldots} \mid \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2) \end{pmatrix}$$

$$E(\nu_3 m_3) = JK\sum_{i=1}^{I}\left(\frac{(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)}{JK} + \frac{(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)}{IJK} - 2\frac{(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)}{IJK}\right)$$

$$E(\nu_3 m_3) = JK\left(\frac{I(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)}{JK} + \frac{I(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)}{IJK} - 2\frac{I(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)}{IJK}\right)$$

$$E(\nu_3 m_3) = \left(I(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2) + (\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2) - 2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)\right)$$

$$E(\nu_3 m_3) = \left(I(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2) - (\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)\right)$$

$$E(\nu_3 m_3) = (\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)(I - 1)$$

$$E(\nu_3 m_3) = (\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)\nu_3 \quad \text{since} \quad \nu_3 = I - 1$$

$$E(m_3) = \frac{(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)\nu_3}{\nu_3} = (\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2).$$

Proof of theorem 6.4.3

We want to derive the posterior distribution of $\mu_i = \mu + r_i \quad (i = 1,\ldots,I)$ given $\underline{Y}$ and the variance components σ_1^2, σ_2^2 and σ_3^2. To do this we derive the posterior distribution of μ_i given $\mu, \sigma_1^2,\ \sigma_2^2, \sigma_3^2$ and $\underline{Y}$ which is normally distributed.

$$Y_{ijk} = \mu + r_i + c_{ij} + \varepsilon_{ijk} \quad i = 1,\ldots,I, \quad j = 1,\ldots,J \quad \text{and} \quad k = 1,\ldots,K.$$

where

$$r_i \sim N(0,\sigma_3^2), \quad c_{ij} \sim N(0,\sigma_2^2) \quad \text{and} \quad \varepsilon_{ijk} \sim N(0,\sigma_1^2).$$

Therefore

$$\mu + r_i \sim N(\mu,\sigma_3^2) \quad \textit{or} \quad \textit{simply} \quad \mu_i \sim N(\mu,\sigma_3^2) \quad \text{since} \quad \mu_i = \mu + r_i$$

The posterior distribution of $\mu_i \mid \underline{Y}, \mu, \sigma_1^2, \sigma_2^2$ is now calculated as follows

$$p(\mu_i \mid \underline{Y}, \mu, \sigma_1^2, \sigma_2^2) \propto \prod_{j=1}^{J}\prod_{k=1}^{K} f(Y_{ijk} \mid \mu_i) p(\mu_i)$$

$$\propto \prod_{j=1}^{J}\prod_{k=1}^{K} \exp(-\frac{1}{2}\frac{(Y_{ijk}-\mu_i)^2}{(\sigma_1^2+\sigma_2^2)}) \exp(-\frac{1}{2}\frac{(\mu_i-\mu)^2}{\sigma_3^2})$$

$$= \exp(-\frac{1}{2}\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk}-\mu_i)^2}{(\sigma_1^2+\sigma_2^2)}) \exp(-\frac{1}{2}\frac{(\mu_i-\mu)^2}{\sigma_3^2})$$

$$= \exp\left(-\frac{1}{2}\left\{\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk}-\mu_i)^2}{(\sigma_1^2+\sigma_2^2)} + \frac{(\mu_i-\mu)^2}{\sigma_3^2}\right\}\right)$$

but

$$\sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk}-\mu_i)^2 = \sum_{j=1}^{J}\sum_{k=1}^{K}\left((Y_{ijk}-\overline{\overline{Y_{i..}}}) + (\overline{\overline{Y_{i..}}}-\mu_i)\right)^2$$

$$= \sum_{j=1}^{J}\sum_{k=1}^{K}\left((Y_{ijk}-\overline{\overline{Y_{i..}}})^2 + (\overline{\overline{Y_{i..}}}-\mu_i)^2 + 2(Y_{ijk}-\overline{\overline{Y_{i..}}})(\overline{\overline{Y_{i..}}}-\mu_i)\right)$$

$$= \sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk}-\overline{\overline{Y_{i..}}})^2 + \sum_{j=1}^{J}\sum_{k=1}^{K}(\overline{\overline{Y_{i..}}}-\mu_i)^2 + 2\sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk}-\overline{\overline{Y_{i..}}})(\overline{\overline{Y_{i..}}}-\mu_i)$$

$$= \sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk}-\overline{\overline{Y_{i..}}})^2 + JK(\overline{\overline{Y_{i..}}}-\mu_i)^2 + 0$$

$$= \sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk}-\overline{\overline{Y_{i..}}})^2 + JK(\overline{\overline{Y_{i..}}}-\mu_i)^2$$

Therefore

$$p(\mu_i \mid \underline{Y}, \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2) \propto \exp\left(-\frac{1}{2}\left\{\frac{\sum_{j=1}^{J}\sum_{k=1}^{K}(Y_{ijk}-\overline{\overline{Y_{i..}}})^2}{(\sigma_1^2+\sigma_2^2)} + \frac{JK(\overline{\overline{Y_{i..}}}-\mu_i)^2}{(\sigma_1^2+K\sigma_2^2)} + \frac{(\mu_i-\mu)^2}{\sigma_3^2}\right\}\right)$$

We omit all factors that involve Y_{ij} but do not depend on μ_i.

Therefore:

$$p(\mu_i \mid \underline{Y}, \mu, \sigma_1^2, \sigma_2^2) \propto \exp\left(-\frac{1}{2}\left\{\frac{JK(\overline{\overline{Y_{i..}}} - \mu_i)^2}{(\sigma_1^2 + K\sigma_2^2)} + \frac{(\mu_i - \mu)^2}{\sigma_3^2}\right\}\right)$$

$$\frac{JK(\overline{\overline{Y_{i..}}} - \mu_i)^2}{(\sigma_1^2 + K\sigma_2^2)} + \frac{(\mu_i - \mu)^2}{\sigma_3^2} = \frac{JK(\overline{\overline{Y_{i..}}}^2 + \mu_i^2 - 2\overline{\overline{Y_{i..}}}\mu_i)}{(\sigma_1^2 + K\sigma_2^2)} + \frac{(\mu_i^2 + \mu^2 - 2\mu_i\mu)}{\sigma_3^2}$$

$$= \mu_i^2\left(\frac{JK}{(\sigma_1^2 + K\sigma_2^2)} + \frac{1}{\sigma_3^2}\right) - 2\mu_i\left(\frac{J\overline{\overline{Y_{i..}}}}{(\sigma_1^2 + K\sigma_2^2)} + \frac{\mu}{\sigma_3^2}\right) + \frac{J\overline{\overline{Y_{i..}}}^2}{(\sigma_1^2 + K\sigma_2^2)} + \frac{\mu^2}{\sigma_3^2}$$

$$= \mu_i^2\left(\frac{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2)\sigma_3^2}\right) - 2\mu_i\left(\frac{(\sigma_1^2 + K\sigma_2^2)\mu + J\overline{\overline{Y_{i..}}}\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2)\sigma_3^2}\right) + \frac{J\overline{\overline{Y_{i..}}}^2}{(\sigma_1^2 + K\sigma_2^2)} + \frac{\mu^2}{\sigma_3^2}$$

$$= \frac{1}{\dfrac{(\sigma_1^2 + K\sigma_2^2)\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2 + JK\sigma^2_3)}}\left[\mu_i^2 - 2\mu_i\left(\frac{(\sigma_1^2 + K\sigma_2^2)\mu + JK\overline{\overline{Y_{i..}}}\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2}\right)\right] + \frac{JK\overline{\overline{Y_{i..}}}^2}{(\sigma_1^2 + K\sigma_2^2)} + \frac{\mu^2}{\sigma_3^2}$$

$$= \frac{1}{\dfrac{(\sigma_1^2 + K\sigma_2^2)\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2 + JK\sigma^2_3)}}\left[\mu_i^2 - 2\mu_i\left(\frac{(\sigma_1^2 + K\sigma_2^2)\mu + JK\overline{\overline{Y_{i..}}}\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2}\right) + \left(\frac{(\sigma_1^2 + K\sigma_2^2)\mu + JK\overline{\overline{Y_{i..}}}\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2}\right)^2\right]$$

$$+ \frac{J\overline{Y_{i.}}^2}{\sigma_1^2} + \frac{\mu^2}{\sigma_2^2} - \left(\frac{(\sigma_1^2 + K\sigma_2^2)\mu + JK\overline{\overline{Y_{i..}}}\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2}\right)^2$$

$$= \frac{1}{\dfrac{(\sigma_1^2 + K\sigma_2^2)\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2 + JK\sigma^2_3)}}\left[\mu_i - \left(\frac{(\sigma_1^2 + K\sigma_2^2)\mu + JK\overline{\overline{Y_{i..}}}\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2}\right)\right]^2 + \textit{other terms}$$

not involving μ_i.

If we drop all terms which do not involve μ_i in the expression for $p(\mu_i \mid \underline{Y}, \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2)$ we get

$$p(\mu_i \mid \underline{Y}, \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2) \propto \exp\left(-\frac{1}{2}\left\{ \frac{\left[\mu_i - \left(\frac{(\sigma_1^2 + K\sigma_2^2)\mu + JK\overline{\overline{Y}}_{i..}\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2}\right)\right]^2}{\frac{(\sigma_1^2 + K\sigma_2^2)\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2 + JK\sigma^2{}_3)}} \right\} \right)$$

This is proportional to a normal distribution with mean $\left(\frac{(\sigma_1^2 + K\sigma_2^2)\mu + JK\sigma_3^2\overline{\overline{Y}}_{i..}}{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2}\right)$ and variance $\frac{(\sigma_1^2 + K\sigma_2^2)\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2 + JK\sigma^2{}_3)}$

$$\textit{i.e.}\quad \mu_i \mid \underline{Y}, \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N\left(\left(\frac{(\sigma_1^2 + K\sigma_2^2)\mu + JK\sigma_3^2\overline{\overline{Y}}_{i..}}{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2}\right), \frac{(\sigma_1^2 + K\sigma_2^2)\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2 + JK\sigma^2{}_3)} \right)$$

We will now derive the posterior distribution of $\mu_i = \mu + r_i \quad (i = 1,\ldots, I)$ given $\underline{Y}$ and the variance components σ_1^2, σ_2^2 and σ_3^2. To do this we firstly appeal to the derived posterior distribution of μ_i given μ, σ_1^2, σ_2^2, σ_3^2 and $\underline{Y}$ which is normal with mean

$$E(\mu_i \mid \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2, \underline{Y}) = \frac{JK\sigma_3^2\overline{\overline{Y}}_{i..}}{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2} + \frac{(\sigma_1^2 + K\sigma_2^2)\mu}{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2}$$

and variance

$$\mathrm{Var}(\mu_i \mid \mu, \sigma_1^2, \sigma_2^2, \sigma^2{}_3, \underline{Y}) = \frac{(\sigma_1^2 + K\sigma_2^2)\sigma_3^2}{(\sigma_1^2 + K\sigma_2^2 + JK\sigma^2{}_3)}$$

Since $\mu \mid Y, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N\left\{ \overline{\overline{\overline{Y}}}_{...}, \frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{IJK} \right\}$, it follows that μ_i given σ_1^2, σ_2^2 and σ_3^2 will be normal with mean

$$E(\mu_i \mid \sigma_1^2, \sigma_2^2, \sigma_3^2, \underline{Y}) = \frac{JK\sigma_3^2\overline{\overline{Y}}_{i..}}{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2} + \frac{(\sigma_1^2 + K\sigma_2^2)}{(\sigma_1^2 + K\sigma_2^2) + JK\sigma_3^2}\overline{\overline{\overline{Y}}}_{...}$$ and variance

$$\mathrm{Var}(\mu_i \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2) = E_\mu\left\{ \mathrm{Var}(\mu_i \mid \underline{Y}, \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2) \right\} + \mathrm{Var}_\mu\left\{ E(\mu_i \mid \underline{Y}, \mu, \sigma_1^2, \sigma_2^2, \sigma_3^2) \right\}$$

$$\mathrm{Var}(\mu_i \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2) = E_\mu \left\{ \frac{(\sigma_1^{\,2} + K\sigma_2^{\,2})\sigma_3^{\,2}}{(\sigma_1^{\,2} + K\sigma_2^{\,2} + JK\sigma^2{}_3)} \right\} + \mathrm{Var}_\mu \left\{ \frac{JK\sigma_3^{\,2}\overline{\overline{Y}}_{i..}}{(\sigma_1^{\,2} + K\sigma_2^{\,2}) + JK\sigma_3^{\,2}} + \frac{(\sigma_1^{\,2} + K\sigma_2^{\,2})\mu}{(\sigma_1^{\,2} + K\sigma_2^{\,2}) + JK\sigma_3^{\,2}} \right\}$$

$$= \frac{(\sigma_1^{\,2} + K\sigma_2^{\,2})\sigma_3^{\,2}}{(\sigma_1^{\,2} + K\sigma_2^{\,2} + JK\sigma^2{}_3)} + \left(\frac{(\sigma_1^{\,2} + K\sigma_2^{\,2})}{(\sigma_1^{\,2} + K\sigma_2^{\,2} + JK\sigma^2{}_3)} \right)^2 \frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{IJK}$$

$$= \frac{(\sigma_1^{\,2} + K\sigma_2^{\,2})\sigma_3^{\,2}}{(\sigma_1^{\,2} + K\sigma_2^{\,2} + JK\sigma^2{}_3)} + \left(\frac{(\sigma_1^{\,2} + K\sigma_2^{\,2})^2}{IJK(\sigma_1^{\,2} + K\sigma_2^{\,2} + JK\sigma^2{}_3)} \right)$$

$$= \frac{(\sigma_1^{\,2} + K\sigma_2^{\,2})}{(\sigma_1^{\,2} + K\sigma_2^{\,2} + JK\sigma^2{}_3)} \left\{ \sigma_3^2 + \frac{(\sigma_1^{\,2} + K\sigma_2^{\,2})}{IJK} \right\}$$

$$= \frac{(\sigma_1^{\,2} + K\sigma_2^{\,2})}{(\sigma_1^{\,2} + K\sigma_2^{\,2} + JK\sigma^2{}_3)} \left\{ \frac{IJK\sigma_3^2 + \sigma_1^{\,2} + K\sigma_2^{\,2}}{IJK} \right\}$$

Proof of theorem 6.5.1

$\sin ce$ $$\mu \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N\left\{ \overline{\overline{\overline{Y}}}_{...}, \frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{IJK} \right\}$$

$$\frac{\mu - l_0}{\left(\frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{JK} \right)^{\frac{1}{2}}} \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N\left(\frac{\overline{\overline{\overline{Y}}}_{...} - l_0}{\left(\frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{JK} \right)^{\frac{1}{2}}}, \quad \frac{1}{I} \right)$$

$$\frac{\mu - l_0}{3\left(\frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{JK} \right)^{\frac{1}{2}}} \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N\left(\frac{\overline{\overline{\overline{Y}}}_{...} - l_0}{\left(\frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{JK} \right)^{\frac{1}{2}}}, \quad \frac{1}{9I} \right)$$

$$\frac{\mu - l_0}{3\left(\frac{(\sigma_{123}{}^2)}{JK}\right)^{\frac{1}{2}}} \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N\left(\frac{\overline{\overline{\overline{Y}}}_{\dots} - l_0}{\left(\frac{\sigma_{123}{}^2}{JK}\right)^{\frac{1}{2}}}, \quad \frac{1}{9I}\right) \sin ce\ \sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2 = \sigma_{123}{}^2$$

$$_3P_{pl}^1 \mid \underline{Y}, \sigma_1^2, \sigma_2^2, \sigma_3^2 \sim N\left(\frac{\overline{\overline{\overline{Y}}}_{\dots} - l_0}{\left(\frac{\sigma_{123}{}^2}{JK}\right)^{\frac{1}{2}}}, \quad \frac{1}{9I}\right)$$

Proof of theorem 6.6.1

The integrated likelihood function is given by

$$L(\mu, \sigma_1^2, \sigma_2^2, \sigma_3^2 \mid \underline{Y}) \propto (\sigma_1^2)^{-\frac{1}{2}\nu_1} (\sigma_1^2 + K\sigma_2^2)^{-\frac{1}{2}\nu_2} (\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^{-\frac{1}{2}(\nu_3+1)} \times$$

$$\exp\left\{-\frac{1}{2}\left[\frac{JK\sum_{i=1}^{I}(\overline{\overline{Y}}_{i\cdot\cdot} - \mu)^2}{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2} + \frac{\nu_2 m_2}{\sigma_1^2 + K\sigma_2^2} + \frac{\nu_1 m_1}{\sigma_1^2}\right]\right\}$$

$$\propto (\sigma_1^2)^{-\frac{1}{2}\nu_1} (\sigma_1^2 + K\sigma_2^2)^{-\frac{1}{2}\nu_2} (\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^{-\frac{1}{2}(\nu_3+1)} \times$$

$$\exp\left\{-\frac{1}{2}\left[\frac{IJK(\overline{\overline{\overline{Y}}}_{\dots} - \mu)^2}{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2} + \frac{\nu_3 m_3}{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2} + \frac{\nu_2 m_2}{\sigma_1^2 + K\sigma_2^2} + \frac{\nu_1 m_1}{\sigma_1^2}\right]\right\}.$$

The Fisher information matrix is obtained by differentiating $\log L(\mu, \sigma_1^2, \sigma_2^2, \sigma_3^2 \mid \underline{Y})$ twice with respect to the unknown parameters and taking minus the expected values.

The Fisher information matrix is given as:

$$F(\mu,\sigma_1^2,\sigma_2^2,\sigma_3^2)=\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix}$$

where

$$L(\mu,\sigma_1^2,\sigma_2^2,\sigma_3^2\,|\,\underline{Y}) \propto (\sigma_1^2)^{-\frac{1}{2}\nu_1}(\sigma_1^2+K\sigma_2^2)^{-\frac{1}{2}\nu_2}(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^{-\frac{1}{2}(\nu_3+1)} \times$$
$$\exp\left\{-\frac{1}{2}\left[\frac{IJK(\overline{\overline{\overline{Y}}}_{\dots}-\mu)^2}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}+\frac{\nu_3 m_3}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}+\frac{\nu_2 m_2}{\sigma_1^2+K\sigma_2^2}+\frac{\nu_1 m_1}{\sigma_1^2}\right]\right\}$$

$$\ln L(\mu,\sigma_1^2,\sigma_2^2,\sigma_3^2\,|\,\underline{Y}) = \ell \propto \ln[(\sigma_1^2)^{-\frac{1}{2}\nu_1}(\sigma_1^2+K\sigma_2^2)^{-\frac{1}{2}\nu_2}(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^{-\frac{1}{2}(\nu_3+1)} \times$$
$$\exp\left\{-\frac{1}{2}\left[\frac{IJK(\overline{\overline{\overline{Y}}}_{\dots}-\mu)^2}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}+\frac{\nu_3 m_3}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}+\frac{\nu_2 m_2}{\sigma_1^2+K\sigma_2^2}+\frac{\nu_1 m_1}{\sigma_1^2}\right]\right\}]$$

$$\ell = -\frac{\nu_1}{2}\ln(\sigma_1^2)-\frac{\nu_2}{2}\ln(\sigma_1^2+K\sigma_2^2)-\frac{(\nu_3+1)}{2}\ln(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)$$
$$-\frac{1}{2}\left[\frac{IJK(\overline{\overline{\overline{Y}}}_{\dots}-\mu)^2}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}+\frac{\nu_3 m_3}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}+\frac{\nu_2 m_2}{\sigma_1^2+K\sigma_2^2}+\frac{\nu_1 m_1}{\sigma_1^2}\right]$$

$$\frac{\partial\ell}{\partial\mu}=\frac{IJK(\overline{\overline{\overline{Y}}}_{\dots}-\mu)}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}$$

$$\frac{\partial\ell}{\partial\mu^2}=-\frac{IJK(\overline{\overline{\overline{Y}}}_{\dots}-\mu)^0}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}=-\frac{IJK}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}$$

since $\nu_3 = I-1$

$$\frac{\partial\ell}{\partial\mu^2}=-\frac{(\nu_3+1)JK}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2} \text{ and } -E\left(\frac{\partial\ell}{\partial\mu^2}\right)=F_{11}=\frac{(\nu_3+1)JK}{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}$$

Further

$$\frac{\partial \ell}{\partial \sigma_1{}^2} = -\frac{\nu_1}{2\sigma_1^2} - \frac{\nu_2}{2(\sigma_1^2 + K\sigma_2^2)} - \frac{(\nu_3+1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)} + \frac{1}{2}\frac{IJK(\overline{\overline{\overline{Y}}}_{\dots} - \mu)^2}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2}$$

$$+\frac{1}{2}\frac{\nu_3 m_3}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} + \frac{1}{2}\frac{\nu_2 m_2}{\left(\sigma_1^2 + K\sigma_2^2\right)^2} + \frac{1}{2}\frac{\nu_1 m_1}{\left(\sigma_1^2\right)^2}$$

$$\frac{\partial^2 \ell}{(\partial \sigma_1{}^2)^2} = \frac{\nu_1}{2\left(\sigma_1^2\right)^2} + \frac{\nu_2}{2(\sigma_1^2 + K\sigma_2^2)^2} + \frac{(\nu_3+1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} - \frac{1}{2}\frac{IJK(\overline{\overline{\overline{Y}}}_{\dots} - \mu)^2}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$-\frac{2}{2}\frac{\nu_3 m_3}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3} - \frac{2}{2}\frac{\nu_2 m_2}{\left(\sigma_1^2 + K\sigma_2^2\right)^3} - \frac{2}{2}\frac{\nu_1 m_1}{\left(\sigma_1^2\right)^3}$$

and we know $E\left(m_1\right) = \sigma_1{}^2$, $E\left(m_2\right) = (\sigma_1{}^2 + K\sigma_2{}^2)$ and $E\left(m_3\right) = (\sigma_1{}^2 + K\sigma_2{}^2 + JK\sigma_3{}^2)$.

Therefore

$$-E\left(\frac{\partial^2 \ell}{(\partial \sigma_1{}^2)^2}\right) = -\frac{\nu_1}{2\left(\sigma_1^2\right)^2} - \frac{\nu_2}{2(\sigma_1^2 + K\sigma_2^2)^2} - \frac{(\nu_3+1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{1}{2}\frac{IJK.E(\overline{\overline{\overline{Y}}}_{\dots} - \mu)^2}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$+\frac{\nu_3 E(m_3)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3} + \frac{\nu_2 E(m_2)}{\left(\sigma_1^2 + K\sigma_2^2\right)^3} + \frac{\nu_1 E(m_1)}{\left(\sigma_1^2\right)^3}$$

$$-E\left(\frac{\partial^2 \ell}{(\partial \sigma_1{}^2)^2}\right) = -\frac{\nu_1}{2\left(\sigma_1^2\right)^2} - \frac{\nu_2}{2(\sigma_1^2 + K\sigma_2^2)^2} - \frac{(\nu_3+1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{1}{2}\frac{IJK\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}{IJK\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$+\frac{\nu_3(\sigma_1{}^2 + K\sigma_2{}^2 + JK\sigma_3{}^2)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3} + \frac{\nu_2(\sigma_1{}^2 + K\sigma_2{}^2)}{\left(\sigma_1^2 + K\sigma_2^2\right)^3} + \frac{\nu_1 \sigma_1{}^2}{\left(\sigma_1^2\right)^3}$$

$$-E\left(\frac{\partial^2 \ell}{(\partial \sigma_1{}^2)^2}\right) = -\frac{\nu_1}{2\left(\sigma_1^2\right)^2} + \frac{\nu_1}{\left(\sigma_1^2\right)^2} - \frac{\nu_2}{2(\sigma_1^2 + K\sigma_2^2)^2} + \frac{\nu_2}{\left(\sigma_1^2 + K\sigma_2^2\right)^2} - \frac{(\nu_3+1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2}$$

$$+\frac{(\nu_3+1)}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2}$$

$$-E\left(\frac{\partial^2 \ell}{(\partial \sigma_1{}^2)^2}\right) = \frac{\nu_1}{2\left(\sigma_1^2\right)^2} + \frac{\nu_2}{2\left(\sigma_1^2 + K\sigma_2^2\right)^2} + \frac{(\nu_3+1)}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2}$$

$$-E\left(\frac{\partial^2 \ell}{(\partial \sigma_1{}^2)^2}\right) = \frac{\nu_1}{2\left(\sigma_1^2\right)^2} + \frac{\nu_2}{2\left(\sigma_1^2 + K\sigma_2^2\right)^2} + \frac{(\nu_3+1)}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2}$$

$$-E\left(\frac{\partial^2\ell}{(\partial\sigma_1^2)^2}\right)=F_{22}=\frac{1}{2}\left\{\frac{\nu_1}{\left(\sigma_1^2\right)^2}+\frac{\nu_2}{\left(\sigma_1^2+K\sigma_2^2\right)^2}+\frac{(\nu_3+1)}{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^2}\right\}$$

Further

$$\frac{\partial\ell}{\partial\sigma_2{}^2}=0-\frac{K\nu_2}{2(\sigma_1^2+K\sigma_2^2)}-\frac{K(\nu_3+1)}{2(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)}+\frac{IJK^2(\overline{\overline{\overline{Y}}}_{...}-\mu)^2}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^2}+\frac{K\nu_3m_3}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^2}$$
$$+\frac{K\nu_2m_2}{2\left(\sigma_1^2+K\sigma_2^2\right)^2}$$

$$\frac{\partial^2\ell}{(\partial\sigma_2^2)^2}=0+\frac{K^2\nu_2}{2(\sigma_1^2+K\sigma_2^2)^2}+\frac{K^2(\nu_3+1)}{2(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2}-\frac{IJK^3(\overline{\overline{\overline{Y}}}_{...}-\mu)^2}{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^3}-\frac{K^2\nu_3m_3}{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^3}$$
$$-\frac{K^2\nu_2m_2}{\left(\sigma_1^2+K\sigma_2^2\right)^3}$$

$$-E\left(\frac{\partial^2\ell}{(\partial\sigma_2{}^2)^2}\right)=0-\frac{K^2\nu_2}{2(\sigma_1^2+K\sigma_2^2)^2}-\frac{K^2(\nu_3+1)}{2(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2}+\frac{IJK^3.E(\overline{\overline{\overline{Y}}}_{...}-\mu)^2}{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^3}+\frac{K^2\nu_3E(m_3)}{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^3}$$
$$+\frac{K^2\nu_2E(m_2)}{\left(\sigma_1^2+K\sigma_2^2\right)^3}$$

$$-E\left(\frac{\partial^2\ell}{(\partial\sigma_2{}^2)^2}\right)=-\frac{K^2\nu_2}{2(\sigma_1^2+K\sigma_2^2)^2}-\frac{K^2(\nu_3+1)}{2(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2}+\frac{IJK^3.\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}{IJK\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^3}+\frac{K^2\nu_3(\sigma_1{}^2+K\sigma_2{}^2+JK\sigma_3{}^2)}{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^3}$$
$$+\frac{K^2\nu_2\left(\sigma_1^2+K\sigma_2^2\right)}{\left(\sigma_1^2+K\sigma_2^2\right)^3}$$

$$-E\left(\frac{\partial^2\ell}{(\partial\sigma_2{}^2)^2}\right)=-\frac{K^2\nu_2}{2(\sigma_1^2+K\sigma_2^2)^2}-\frac{K^2(\nu_3+1)}{2(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2}+\frac{K^2}{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^2}+\frac{K^2\nu_3}{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^2}+\frac{K^2\nu_2}{\left(\sigma_1^2+K\sigma_2^2\right)^2}$$

$$-E\left(\frac{\partial^2\ell}{(\partial\sigma_2{}^2)^2}\right)=-\frac{K^2\nu_2}{2(\sigma_1^2+K\sigma_2^2)^2}+\frac{K^2\nu_2}{\left(\sigma_1^2+K\sigma_2^2\right)^2}-\frac{K^2(\nu_3+1)}{2(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2}+\frac{K^2(\nu_3+1)}{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^2}$$

$$-E\left(\frac{\partial^2\ell}{(\partial\sigma_2{}^2)^2}\right)=\frac{K^2\nu_2}{2(\sigma_1^2+K\sigma_2^2)^2}+\frac{K^2(\nu_3+1)}{2(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2}$$

$$-E\left(\frac{\partial^2\ell}{(\partial\sigma_2{}^2)^2}\right)=F_{33}=\frac{K^2}{2}\left\{\frac{\nu_2}{(\sigma_1^2+K\sigma_2^2)^2}+\frac{(\nu_3+1)}{(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2}\right\}$$

Further

$$\frac{\partial \ell}{\partial {\sigma_3}^2} = 0 + 0 - \frac{JK(\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)} + \frac{I\left(JK\right)^2 (\overline{\overline{\overline{Y}}}_{\cdots} - \mu)^2}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} + \frac{JK\nu_3 m_3}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2}$$

$$\frac{\partial^2 \ell}{\left(\partial {\sigma_3}^2\right)^2} = \frac{\left(JK\right)^2 (\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} - \frac{I\left(JK\right)^3 (\overline{\overline{\overline{Y}}}_{\cdots} - \mu)^2}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3} - \frac{\left(JK\right)^2 \nu_3 m_3}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$-E\left(\frac{\partial^2 \ell}{\left(\partial {\sigma_3}^2\right)^2}\right) = -\frac{\left(JK\right)^2 (\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{I\left(JK\right)^3 E(\overline{\overline{\overline{Y}}}_{\cdots} - \mu)^2}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3} + \frac{\left(JK\right)^2 \nu_3 E(m_3)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$-E\left(\frac{\partial^2 \ell}{\left(\partial {\sigma_3}^2\right)^2}\right) = -\frac{\left(JK\right)^2 (\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{I\left(JK\right)^3 \left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}{IJK\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3} + \frac{\left(JK\right)^2 \nu_3 \left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$-E\left(\frac{\partial^2 \ell}{\left(\partial {\sigma_3}^2\right)^2}\right) = -\frac{\left(JK\right)^2 (\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{\left(JK\right)^2}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} + \frac{\left(JK\right)^2 \nu_3}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2}$$

$$-E\left(\frac{\partial^2 \ell}{\left(\partial {\sigma_3}^2\right)^2}\right) = F_{44} = -\frac{\left(JK\right)^2 (\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{\left(JK\right)^2 (\nu_3 + 1)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} = \frac{\left(JK\right)^2 (\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2}$$

Also

$$\frac{\partial \ell}{\partial \mu \partial \sigma_1^2} = -\frac{IJK(\overline{\overline{\overline{Y}}}_{\cdots} - \mu)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} \qquad \therefore -E\left(\frac{\partial \ell}{\partial \mu \partial \sigma_1^2}\right) = F_{12} = F_{21} = \frac{IJKE(\overline{\overline{\overline{Y}}}_{\cdots} - \mu)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} = 0$$

$$\frac{\partial \ell}{\partial \mu \partial \sigma_2^2} = -\frac{IJK^2(\overline{\overline{\overline{Y}}}_{\cdots} - \mu)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} \qquad \therefore -E\left(\frac{\partial \ell}{\partial \mu \partial \sigma_2^2}\right) = F_{13} = F_{31} = \frac{IJK^2 E(\overline{\overline{\overline{Y}}}_{\cdots} - \mu)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} = 0$$

$$\frac{\partial \ell}{\partial \mu \partial \sigma_3^2} = -\frac{IJ^2K^2(\overline{\overline{\overline{Y}}}_{\cdots} - \mu)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} \qquad \therefore -E\left(\frac{\partial \ell}{\partial \mu \partial \sigma_3^2}\right) = F_{14} = F_{41} = \frac{IJ^2K^2 E(\overline{\overline{\overline{Y}}}_{\cdots} - \mu)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} = 0$$

Further

$$\frac{\partial^2 \ell}{\partial\sigma_1^2 \partial\sigma_2^2} = -0 + \frac{K\nu_2}{2(\sigma_1^2 + K\sigma_2^2)^2} + \frac{K(\nu_3+1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} - \frac{IJK^2(\overline{\overline{\overline{Y}}}_{\dots} - \mu)^2}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$- \frac{K\nu_3 m_3}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3} - \frac{K\nu_2 m_2}{\left(\sigma_1^2 + K\sigma_2^2\right)^3}$$

$$-E\left(\frac{\partial^2 \ell}{\partial\sigma_1^2 \partial\sigma_2^2}\right) = -\frac{K\nu_2}{2(\sigma_1^2 + K\sigma_2^2)^2} - \frac{K(\nu_3+1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{IJK^2.E(\overline{\overline{\overline{Y}}}_{\dots} - \mu)^2}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$+ \frac{K\nu_3 E(m_3)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3} + \frac{K\nu_2 E(m_2)}{\left(\sigma_1^2 + K\sigma_2^2\right)^3}$$

$$-E\left(\frac{\partial^2 \ell}{\partial\sigma_1^2 \partial\sigma_2^2}\right) = -\frac{K\nu_2}{2(\sigma_1^2 + K\sigma_2^2)^2} - \frac{K(\nu_3+1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{IJK^2.\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}{IJK\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$+ \frac{K\nu_3\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3} + \frac{K\nu_2\left(\sigma_1^2 + K\sigma_2^2\right)}{\left(\sigma_1^2 + K\sigma_2^2\right)^3}$$

$$-E\left(\frac{\partial^2 \ell}{\partial\sigma_1^2 \partial\sigma_2^2}\right) = -\frac{K\nu_2}{2(\sigma_1^2 + K\sigma_2^2)^2} - \frac{K(\nu_3+1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{K}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2}$$

$$+ \frac{K\nu_3}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} + \frac{K\nu_2}{\left(\sigma_1^2 + K\sigma_2^2\right)^2}$$

$$-E\left(\frac{\partial^2 \ell}{\partial\sigma_1^2 \partial\sigma_2^2}\right) = -\frac{K\nu_2}{2(\sigma_1^2 + K\sigma_2^2)} + \frac{K\nu_2}{\left(\sigma_1^2 + K\sigma_2^2\right)^2} - \frac{K(\nu_3+1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)} + \frac{K(\nu_3+1)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2}$$

$$-E\left(\frac{\partial^2 \ell}{\partial\sigma_1^2 \partial\sigma_2^2}\right) = \frac{K\nu_2}{2(\sigma_1^2 + K\sigma_2^2)} + \frac{K(\nu_3+1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)}$$

$$-E\left(\frac{\partial^2 \ell}{\partial\sigma_1^2 \partial\sigma_2^2}\right) = F_{23} = F_{32} = \frac{K}{2}\left\{\frac{\nu_2}{(\sigma_1^2 + K\sigma_2^2)} + \frac{(\nu_3+1)}{(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)}\right\}$$

Further

$$\frac{\partial \ell}{\partial {\sigma_1}^2 \partial {\sigma_3}^2} = 0 + 0 + \frac{JK(\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} - \frac{I\left(JK\right)^2 (\overline{\overline{\overline{Y}}}_{\dots} - \mu)^2}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$- \frac{JK\nu_3 m_3}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$-E\left(\frac{\partial \ell}{\partial {\sigma_1}^2 \partial {\sigma_3}^2}\right) = -\frac{JK(\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{I\left(JK\right)^2 E(\overline{\overline{\overline{Y}}}_{\dots} - \mu)^2}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$+ \frac{JK\nu_3 E\left(m_3\right)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$-E\left(\frac{\partial \ell}{\partial {\sigma_1}^2 \partial {\sigma_3}^2}\right) = -\frac{JK(\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{I\left(JK\right)^2 \left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}{IJK\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$+ \frac{JK\nu_3\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$-E\left(\frac{\partial \ell}{\partial {\sigma_1}^2 \partial {\sigma_3}^2}\right) = F_{24} = F_{42} = -\frac{JK(\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{JK(\nu_3 + 1)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} = \frac{JK(\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2}$$

Also

$$\frac{\partial^2 \ell}{\partial {\sigma_2}^2 \partial {\sigma_3}^2} = \frac{JK^2(\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} - \frac{IJ^2K^3(\overline{\overline{\overline{Y}}}_{\dots} - \mu)^2}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3} - \frac{JK^2\nu_3 m_3}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$-E\left(\frac{\partial^2 \ell}{\partial {\sigma_2}^2 \partial {\sigma_3}^2}\right) = -\frac{JK^2(\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{IJ^2K^3E(\overline{\overline{\overline{Y}}}_{\dots} - \mu)^2}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3} + \frac{JK^2\nu_3 E(m_3)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$-E\left(\frac{\partial^2 \ell}{\partial {\sigma_2}^2 \partial {\sigma_3}^2}\right) = -\frac{JK^2(\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{IJ^2K^3\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}{IJK\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3} + \frac{JK^2\nu_3\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^3}$$

$$-E\left(\frac{\partial^2 \ell}{\partial {\sigma_2}^2 \partial {\sigma_3}^2}\right) = F_{34} = F_{43} = -\frac{JK^2(\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2} + \frac{JK^2(\nu_3 + 1)}{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^2} = \frac{JK^2(\nu_3 + 1)}{2(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)^2}$$

The inverse of the Fisher information matrix is given by

$$F^{-1}(\mu,\sigma_1^2,\sigma_2^2,\sigma_3^2) = F^{-1}(\underline{\theta}) = \begin{bmatrix} F^{11} & F^{12} & F^{13} & F^{14} \\ F^{21} & F^{22} & F^{23} & F^{24} \\ F^{31} & F^{32} & F^{33} & F^{34} \\ F^{41} & F^{42} & F^{43} & F^{44} \end{bmatrix}$$

where $F^{11} = \dfrac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{(\nu_3+1)JK}$

$F^{12} \quad = \qquad F^{21} = 0; F^{13} = F^{31} = 0; F^{14} = F^{41} = 0;$

$F^{22} \quad = \quad \dfrac{1}{|H|}\dfrac{\nu_2(\nu_3+1)J^2K^4}{4(\sigma_1^2+K\sigma_2^2)^2(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2};$

$$F^{23} = \frac{-1}{|H|}\frac{J^2K^3\nu_2(\nu_3+1)}{4(\sigma_1^2+K\sigma_2^2)^2(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2} = F^{32};$$

$F^{24} = 0 = F^{42};$

$$F^{33} = \frac{J^2K^2(\nu_3+1)}{4|H|(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2}\left\{\frac{\nu_1}{(\sigma_1^2)^2} + \frac{\nu_2}{(\sigma_1^2+K\sigma_2^2)^2}\right\};$$

$$F^{34} = \frac{-\nu_1(\nu_3+1)JK^2}{4|H|(\sigma_1^2)(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2} = F^{43};$$

$$F^{44} = \frac{\nu_1K^2}{4|H|(\sigma_1^2)^2}\left\{\frac{\nu_2}{(\sigma_1^2+K\sigma_2^2)^2} + \frac{(\nu_3+1)}{(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2}\right\}$$

and

$$|H| = \frac{\nu_1\nu_2(\nu_3+1)J^2K^4}{8(\sigma_1^2)^2(\sigma_1^2+K\sigma_2^2)^2(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2}$$

We are interested in the probability matching prior for (${}_3P^1_{pl}$), the lower process performance index.

Let $\underline{\theta} = [\mu,\sigma_3{}^2,\sigma_2{}^2,\sigma_1{}^2]'$. The capability index is

$$ {}_{3}P_{pl}^{1} = \frac{\mu - l_0}{3\left(\frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{JK}\right)^{\frac{1}{2}}} = \frac{(\mu - l_0)(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{1}{2}}}. $$

Therefore

$$ \frac{\partial t(\underline{\theta})}{\partial \mu} = \frac{(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{1}{2}}},\quad \frac{\partial t(\underline{\theta})}{\partial \sigma_1^{\ 2}} = \frac{-(\mu - l_0)(JK)^{\frac{1}{2}}}{6\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{3}{2}}} $$

$$ \frac{\partial t(\underline{\theta})}{\partial \sigma_2^{\ 2}} = \frac{-(\mu - l_0)J^{\frac{1}{2}}K^{\frac{3}{2}}}{6\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{3}{2}}} \text{ and } \frac{\partial t(\underline{\theta})}{\partial \sigma_3^{\ 2}} = \frac{-(\mu - l_0)(JK)^{\frac{3}{2}}}{6\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{3}{2}}} $$

As mentioned

$$ \nabla'_t(\tilde{\theta}) = \begin{bmatrix} \frac{\partial t(\underline{\theta})}{\partial \mu} & \frac{\partial t(\underline{\theta})}{\partial \sigma_1^{\ 2}} & \frac{\partial t(\underline{\theta})}{\partial \sigma_2^{\ 2}} & \frac{\partial t(\underline{\theta})}{\partial \sigma_3^{\ 2}} \end{bmatrix} = $$

$$ \begin{bmatrix} \frac{(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{1}{2}}} & \frac{-(\mu - l_0)(JK)^{\frac{1}{2}}}{6\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{3}{2}}} & \frac{-(\mu - l_0)J^{\frac{1}{2}}K^{\frac{3}{2}}}{6\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{3}{2}}} & \frac{-(\mu - l_0)(JK)^{\frac{3}{2}}}{6\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{3}{2}}} \end{bmatrix} $$

$$ \nabla'_t(\underline{\theta}) = \frac{(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{1}{2}}} \begin{bmatrix} 1 & \frac{-(\mu - l_0)}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)} & \frac{-(\mu - l_0)K}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)} & \frac{-(\mu - l_0)(JK)}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)} \end{bmatrix} $$

Further

$$ \nabla'_t(\underline{\theta})F^{-1}(\underline{\theta}) = \frac{(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{1}{2}}} \begin{bmatrix} \frac{\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2}{(\nu_3 + 1)JK} & 0 & 0\ \frac{-(\mu - l_0)(JK)}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)} \times \frac{\nu_1 K^2}{4|H|(\sigma_1^2)^2} \frac{\nu_2}{(\sigma_1^2 + K\sigma_2^2)^2} \end{bmatrix} $$

$$\text{with}\left|H\right| = \frac{\nu_1 \nu_2 (\nu_3+1) J^2 K^4}{8(\sigma_1^2)^2(\sigma_1^2+K\sigma_2^2)^2(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2}$$

$$\nabla'_t(\underline{\theta})F^{-1}(\underline{\theta}) = \frac{(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^{\frac{1}{2}}}\left[\frac{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}{(\nu_3+1)JK} \quad 0 \quad 0 \quad \frac{-(\mu-l_0)(JK)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\times\frac{2(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)^2}{(\nu_3+1)J^2K^2}\right]$$

$$\nabla'_t(\underline{\theta})F^{-1}(\underline{\theta}) = \frac{(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^{\frac{1}{2}}}\left[\frac{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}{(\nu_3+1)JK} \quad 0 \quad 0 \quad \frac{-(\mu-l_0)(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)}{(\nu_3+1)(JK)}\right]$$

$$\nabla'_t(\underline{\theta})F^{-1}(\underline{\theta})\nabla_t(\underline{\theta}) = \frac{(JK)}{9\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\left[\frac{\sigma_1^2+K\sigma_2^2+JK\sigma_3^2}{(\nu_3+1)JK} \quad 0 \quad 0 \quad \frac{-(\mu-l_0)(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)}{(\nu_3+1)(JK)}\right]\begin{bmatrix} 1 \\ \frac{-(\mu-l_0)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)} \\ \frac{-(\mu-l_0)K}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)} \\ \frac{-(\mu-l_0)(JK)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)} \end{bmatrix}$$

$$= \frac{(JK)}{9\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\left(\frac{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}{(\nu_3+1)JK}+\frac{-(\mu-l_0)(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)}{(\nu_3+1)(JK)}\times\frac{-(\mu-l_0)(JK)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right)$$

$$= \frac{(JK)}{9\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\left(\frac{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}{(\nu_3+1)JK}+\frac{(\mu-l_0)^2(JK)}{2(\nu_3+1)(JK)}\right)$$

$$= \frac{1}{9}\left(\frac{1}{(\nu_3+1)}+\frac{(\mu-l_0)^2(JK)}{2(\nu_3+1)\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right)$$

$$= \left(\frac{1}{9(\nu_3+1)}+\frac{(\mu-l_0)^2(JK)}{18(\nu_3+1)\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right)$$

$$= \left(\frac{1}{9I}+\frac{(\mu-l_0)^2(JK)}{18I\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right) \text{ since } I = \nu_3+1$$

$$\nabla'_t(\underline{\theta})F^{-1}(\underline{\theta})\nabla_t(\underline{\theta}) = \frac{1}{3I^{\frac{1}{2}}}\left(1+\frac{(\mu-l_0)^2(JK)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right)^{\frac{1}{2}}$$

Define as before

$$\eta(\underline{\theta}) = \frac{\nabla'_t(\underline{\theta})(\underline{\theta})F^{-1}(\underline{\theta})}{\sqrt{\nabla'_t(\underline{\theta})F^{-1}(\underline{\theta})\nabla_t(\underline{\theta})}} = [\eta_1(\underline{\theta}) \quad \eta_2(\underline{\theta}) \quad \eta_3(\underline{\theta}) \quad \eta_4(\underline{\theta})]$$

where

$$\nabla'_t(\underline{\theta})F^{-1}(\underline{\theta}) = \frac{(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^{\frac{1}{2}}}\left[\frac{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}{(\nu_3+1)JK} \quad 0 \quad 0 \quad \frac{-(\mu-l_0)(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)}{(\nu_3+1)(JK)}\right]$$

$$\nabla'_t(\underline{\theta})F^{-1}(\underline{\theta}) = \frac{(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^{\frac{1}{2}}}\left[\frac{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}{(IJK)} \quad 0 \quad 0 \quad \frac{-(\mu-l_0)(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2)}{(IJK)}\right]$$

since $I = \nu_3 + 1$

and

$$\nabla'_t(\underline{\theta})F^{-1}(\underline{\theta})\nabla_t(\underline{\theta}) = \frac{1}{3I^{\frac{1}{2}}}\left(1+\frac{(\mu-l_0)^2(JK)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right)^{\frac{1}{2}}$$

giving

$$\eta_1(\underline{\theta}) = \frac{(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^{\frac{1}{2}}}\frac{\dfrac{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}{(IJK)}}{\dfrac{1}{3I^{\frac{1}{2}}}\left(1+\dfrac{(\mu-l_0)^2(JK)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right)^{\frac{1}{2}}}$$

$$\eta_1(\underline{\theta}) = \frac{(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^{\frac{1}{2}}}3I^{\frac{1}{2}}\frac{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}{(IJK)}\left(1+\frac{(\mu-l_0)^2(JK)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right)^{-\frac{1}{2}}$$

$$\eta_1(\underline{\theta}) = \frac{\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{1}{2}}}{\left(IJK\right)^{\frac{1}{2}}} \left(1 + \frac{(\mu - l_0)^2 (JK)}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}\right)^{-\frac{1}{2}}$$

$\eta_2(\underline{\theta}) = 0$ and $\eta_3(\underline{\theta}) = 0$

$$\eta_4(\underline{\theta}) = \frac{(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{1}{2}}} \frac{\frac{-(\mu - l_0)(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)}{(IJK)}}{\frac{1}{3I^{\frac{1}{2}}}\left(1 + \frac{(\mu - l_0)^2 (JK)}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}\right)^{\frac{1}{2}}}$$

$$\eta_4(\underline{\theta}) = \frac{(JK)^{\frac{1}{2}}}{3\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{1}{2}}} 3I^{\frac{1}{2}} \frac{-(\mu - l_0)(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2)}{(IJK)} \left(1 + \frac{(\mu - l_0)^2 (JK)}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}\right)^{-\frac{1}{2}}$$

$$\eta_4(\underline{\theta}) = \frac{-(\mu - l_0)\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{\frac{1}{2}}}{\left(IJK\right)^{\frac{1}{2}}} \left(1 + \frac{(\mu - l_0)^2 (JK)}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}\right)^{-\frac{1}{2}}$$

For a prior $\pi(\underline{\theta})$ to be a probability-matching prior, the differential equation

$$\sum_{\alpha=1}^{m} \frac{\partial}{\partial \theta_\alpha} \{\eta_\alpha(\underline{\theta})\pi(\underline{\theta})\} = 0$$

$$\frac{\partial}{\partial \mu}\{\eta_1(\underline{\theta})\pi(\underline{\theta})\} + \frac{\partial}{\partial \sigma_1^{\ 2}}\{\eta_2(\underline{\theta})\pi(\underline{\theta})\} + \frac{\partial}{\partial \sigma_2^{\ 2}}\{\eta_3(\underline{\theta})\pi(\underline{\theta})\} + \frac{\partial}{\partial \sigma_3^{\ 2}}\{\eta_4(\underline{\theta})\pi(\underline{\theta})\} = 0$$

must be satisfied.

The probability matching prior is

$$\pi(\underline{\theta}) = \pi(\mu, \sigma_1^{\ 2}, \sigma_2^{\ 2}, \sigma_3^2) \propto \sigma_1^{-2} \left(\sigma_1^2 + K\sigma_2^2\right)^{-1} \left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)^{-\frac{1}{2}} \left(1 + \frac{(\mu - l_0)^2 (JK)}{2\left(\sigma_1^2 + K\sigma_2^2 + JK\sigma_3^2\right)}\right)^{\frac{1}{2}}$$

since

$$\frac{\partial}{\partial\mu}\left\{\frac{\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^{\frac{1}{2}}}{(IJK)^{\frac{1}{2}}}\left(1+\frac{(\mu-l_0)^2(JK)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right)^{-\frac{1}{2}}\times\sigma_1^{-2}\left(\sigma_1^2+K\sigma_2^2\right)^{-1}\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^{-\frac{1}{2}}\left(1+\frac{(\mu-l_0)^2(JK)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right)^{\frac{1}{2}}\right\}+0+0+$$

$$\frac{\partial}{\partial\sigma_3{}^2}\left\{\frac{-(\mu-l_0)\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^{\frac{1}{2}}}{(IJK)^{\frac{1}{2}}}\left(1+\frac{(\mu-l_0)^2(JK)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right)^{-\frac{1}{2}}\times\sigma_1^{-2}\left(\sigma_1^2+K\sigma_2^2\right)^{-1}\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)^{-\frac{1}{2}}\left(1+\frac{(\mu-l_0)^2(JK)}{2\left(\sigma_1^2+K\sigma_2^2+JK\sigma_3^2\right)}\right)^{\frac{1}{2}}\right\}$$

$$\frac{\partial}{\partial\mu}\left\{\frac{1}{(IJK)^{\frac{1}{2}}}\times\sigma_1^{-2}\left(\sigma_1^2+K\sigma_2^2\right)^{-1}\right\}+0+0+\frac{\partial}{\partial\sigma_3{}^2}\left\{\frac{-(\mu-l_0)}{(IJK)^{\frac{1}{2}}}\times\sigma_1^{-2}\left(\sigma_1^2+K\sigma_2^2\right)^{-1}\right\}$$

$$0+0+0+0=0$$

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