APPROXIMATING THE GENERALIZED BURR-GAMMA WITH A GENERALIZED PARETO-TYPE OF DISTRIBUTION

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ABSTRACT

In this paper the Generalized Burr-Gamma (GBG) distribution is considered to model data that includes extreme values. Since the tail of the distribution is often the only interest it is shown in this paper that the tail of the GBG can be approximated by a Generalized Pareto-type (GP-type) of distribution. This approximation is simpler to work with since it only has one parameter, namely the extreme value index (EVI), where the GBG has four parameters. The GP-type of distribution differs from the Generalized Pareto distribution since it is dependent on the threshold t, as is shown in equation (3). This aspect is advantageous in selecting an optimum threshold. The goodness of fit of this approximated distribution is tested by comparing the tail probabilities of the approximated distribution with the tail probabilities of the GBG. The Jeffreys prior of the EVI under the GP-type is derived and the posterior predictive survival distribution is used for predicting high quantiles. A valuable contribution is made on how to select the optimum threshold when working with extreme values.

KEYWORDS: GBG, GP-type, threshold, posterior predictive survival distribution, tail probabilities, high quantiles.

1 INTRODUCTION

In extreme value analysis, the Peaks Over Threshold method became a popular method to predict high quantiles or estimate tail probabilities. Although parametric models exist to model all the data such as the Burr, Frechét, t ,F and others, the generalized Burr-Gamma class is another class of distributions to fit the whole data set (Beirlant et al. 1999). The Generalized Burr-Gamma is fairly flexible since it consists of four parameters. Since the tail is sometimes the only interest we suppose that the data is Generalized Burr-Gamma distributed and for large values of the threshold we approximate the distribution of the tail with a Generalized Pareto-type of distribution. This approximated distribution is dependent on the threshold t and therefore a valuable contribution can be made in selecting an optimum threshold. Further, we explore the estimation of tail probabilities through the posterior predictive survival distribution of the approximated Generalized Pareto-type distribution and the goodness of the approximated Generalized Pareto-type distribution is tested by comparing the tail probabilities with the tail probabilities of the Generalized Burr-Gamma distribution.

2 THE GENERALIZED BURR-GAMMA DISTRIBUTION (GBG)

The GBG class of distributions includes many of the well known extreme value distributions, such as the Gumbel, Weibull, Burr, Generalized Extreme Value and Generalized Pareto distributions to name a few. The GBG is a fairly flexible distribution which contains four parameters, k,μ,σ,ξ, where ξ is known as the extreme value index. If $\xi = 0$ then μ is the mean of $Y = -\log(X)$, where X is GBG distributed. Similarly if $\xi = 0$ then σ is the standard deviation of Y.

The GBG distribution models all the data, also the data in the tail and is given as follows: (Beirlant *et al*. 1999). A random variable X is GBG(k,μ,σ,ξ) distributed when the distribution function is given by

$$F(x) = P(X \le x) = \frac{1}{\Gamma(k)} \int_0^{\upsilon_\xi(x)} e^{-u} u^{k-1} du \tag{1}$$

where

$$\upsilon_\xi(x) = \frac{1}{\xi} \log\left(1 + \xi \nu(x)\right) > 0 \tag{2}$$

and

$$\nu(x) = e^{\left\{\psi(k) + \frac{\log x + \mu}{\sigma}\sqrt{\psi'(k)}\right\}}$$

for

$$1 + \xi \nu(x) > 1.$$

$\psi(k) = \frac{\partial}{\partial k} \log \Gamma(k)$ and $\psi'(k) = \frac{\partial}{\partial k} \psi(k)$ represent the digamma and trigamma functions respectively.

The parameter space is defined as $\Omega = \left\{-\infty < \mu < \infty, \sigma > 0, k > 0, -\infty < \xi < \infty\right\}$.

A very important characteristic shown by Beirlant *et al*. 1999, p. 115 is that $V_\xi \sim \mathrm{GAM}(k,1)$.

3 APPROXIMATING THE GBG TAIL

The following theorem shows how the tail of the GBG above a reasonable high threshold, can be approximated though a Generalized Pareto-type of distribution.

Theorem 1:

If V_ξ is Gamma distributed, then for large t, $V = e^{\left\{\psi(k)+\frac{\log X+\mu}{\sigma}\sqrt{\psi'(k)}\right\}}$ > t is distributed as a Generalized Pareto-type of distribution with survival function given by

$$P\left(V-t>v\,|\,V>t\right)=\left(1+\frac{\xi v}{1+\xi t}\right)^{\frac{-1}{\xi}},\; v>0;\sigma_t,\xi>0, i=1,\dots,N_t. \tag{3}$$

Proof:

From (2), $V=\frac{e^{\xi V_\xi}-1}{\xi}$ and (3) can be rewritten as

$$P\left(V-t>v\,|\,V>t\right)=P\left(\frac{e^{\xi V_\xi}-1}{\xi}>t+v\left|\frac{e^{\xi V_\xi}-1}{\xi}>t\right.\right),$$
$$=\frac{P\left(V_\xi>\frac{\log\left(\xi\left(t+v\right)+1\right)}{\xi}\right)}{P\left(V_\xi>\frac{\log\left(\xi t+1\right)}{\xi}\right)}. \tag{4}$$

Let $\frac{\log\left(\xi\left(t+v\right)+1\right)}{\xi}=a$ and $\frac{\log\left(\xi t+1\right)}{\xi}=b,$ then

$$P\left(V-t>v\,|\,V>t\right)=\frac{\Gamma\left(k,a\right)}{\Gamma\left(k,b\right)} \tag{5}$$

can be expressed as a ratio of two incomplete Gamma functions. The incomplete Gamma function is given by the integral

$$\Gamma(k,a)=\int_a^\infty h^{k-1}e^{-h}dh\,. \tag{6}$$

The following equation is an approximation of the incomplete gamma function for large values of *a* (Amore, 2005)

$$\Gamma(k,a)\approx e^{-a}(1+a)^{k-1}. \tag{7}$$

Thus equation (4) can be expressed as follows

$$\begin{aligned}\frac{\Gamma(k,a)}{\Gamma(k,b)}&=\frac{(1+\xi t+\xi v)^{\frac{-1}{\xi}}}{(1+\xi t)^{\frac{-1}{\xi}}}\frac{\left\{1+\frac{1}{\xi}\left[\ln(1+\xi t+\xi v)\right]\right\}^{k-1}}{\left\{1+\frac{1}{\xi}\left[\ln(1+\xi t)\right]\right\}^{k-1}}\\&=\left(1+\frac{\xi v}{1+\xi t}\right)^{-\frac{1}{\xi}}\left(\frac{1+\frac{1}{\xi}\left[\ln(1+\xi t+\xi v)\right]}{1+\frac{1}{\xi}\left[\ln(1+\xi t)\right]}\right)^{k-1}.\end{aligned} \tag{8}$$

When using L'Hospital's rule (Salas *et al*. 1999), $\left(\frac{1+\frac{1}{\xi}\left[\ln(1+\xi t+\xi v)\right]}{1+\frac{1}{\xi}\left[\ln(1+\xi t)\right]}\right)^{k-1}\to 1$, as $t\to\infty$.

Therefore we conclude that for large *t*, $\frac{\Gamma(k,a)}{\Gamma(k,b)}=\left(1+\frac{\xi v}{1+\xi t}\right)^{-\frac{1}{\xi}}$,which is the distribution function of the Generalized Pareto distribution with parameter ξ on the exceedances above *t*. Thus, for large values of the threshold the GBG distribution can be approximated with the Generalized Pareto-type of distribution with one parameter ξ.

▲

Theorem 1 shows that for a large threshold t, $V > t$ is approximated by the Generalized Pareto-type of distribution, given by the equation

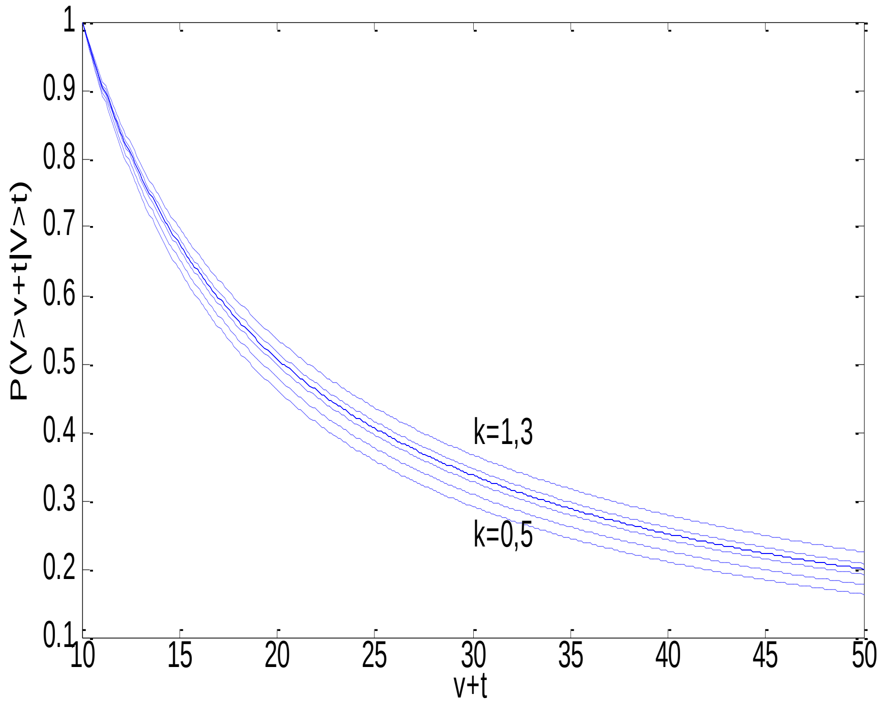
$$\bar{F}(v+t\,|\,t)=\left\{1+\frac{\xi(v+t)}{1+\xi t}\right\}^{\frac{-1}{\xi}} \tag{9}$$

and the incomplete gamma integral is given by the equation

$$\Gamma(k,a)=\int_a^\infty h^{k-1}e^{-h}dh \quad . \tag{10}$$

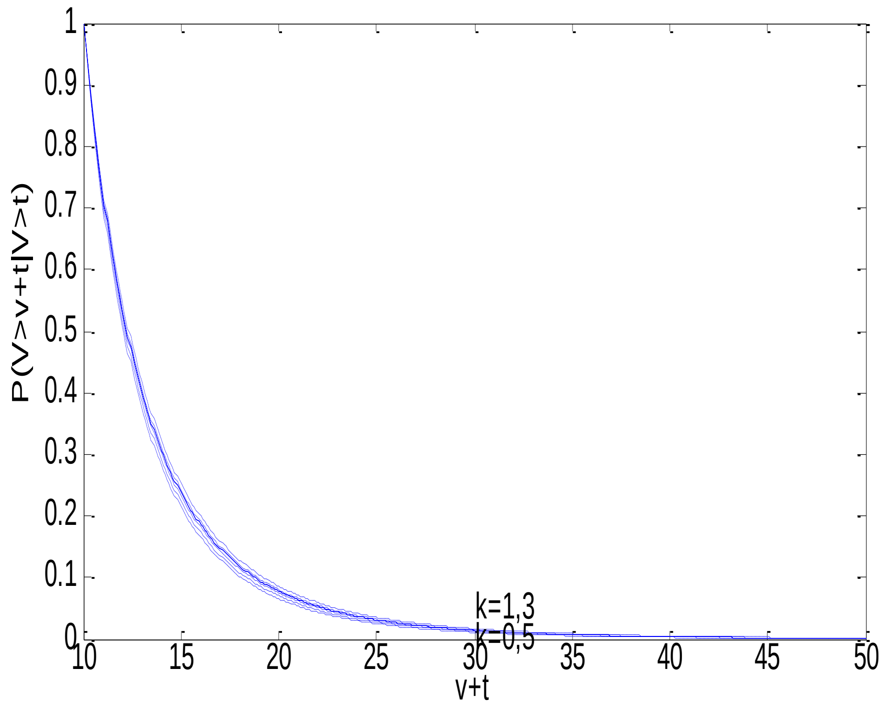
The following Figures show how the approximated Generalized Pareto-type of distribution (9) fits the data above the threshold when compared with the ratio of the incomplete gamma distributions (10). In Figure 1 a threshold is chosen at t = 10, ξ is chosen as 0.95 and k takes on different values between 0.5 and 1.3.

Fig. 1 Comparison between the approximated GPD and the incomplete gamma distribution



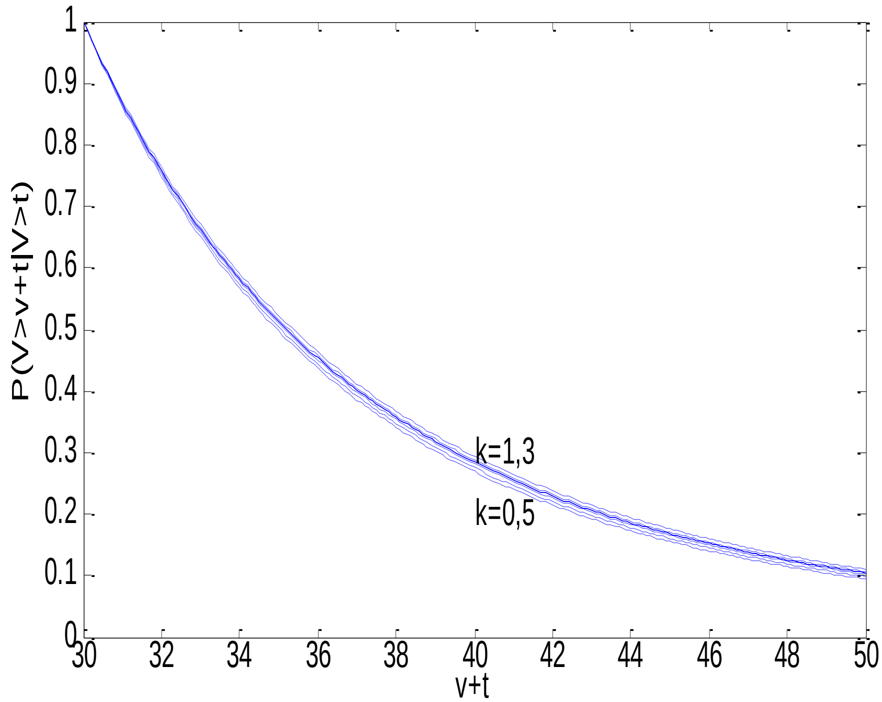
In Figure 2 a threshold is again chosen at t = 10, but now ξ is chosen as $\xi = 0,2$ and k is again chosen as different values between 0,5 and 1,3.

Fig. 2 **Comparison between the approximated GPD and the incomplete gamma distribution with a smaller value of ξ**



In Figure 3 a threshold is chosen at a larger value t = 30, ξ is chosen again as $\xi = 0,2$ and k is chosen again as different values between 0,5 and 1,3.

Fig. 3 **Comparison between the approximated GPD and the incomplete gamma integral with a larger value of *t***



From the above figures it can be seen that, for small values of ξ**,** the approximated Generalized Pareto-type of distribution follows the ratio of the incomplete gamma distributions more closely. If ξ becomes large, close to 1, a higher threshold should be chosen to make sure that the second term of equation (8) strives to 1.

4 FORCASTING TAIL PROBABILITIES USING THE POSTERIOR PREDICTIVE SURVIVAL DISTRIBUTION

In this section tail probabilities of future V_0 values is calculated by using the posterior predictive survival distribution given by

$$P(V_0 > v_0|v,t) = \int \Pi(\xi|v)\left(1+\frac{\xi(v_0+t)}{1+\xi t}\right)^{\frac{-1}{\xi}} d\xi\,, \quad 0 > \xi < \infty \tag{11}$$

where $\left(1+\frac{\xi v_0}{1+\xi t}\right)^{\frac{-1}{\xi}}$ is the approximated Generalized Pareto-type distribution of the V_0 values above a large enough threshold and $\Pi(\xi|v)$ is the posterior distribution which is proportional to the prior times the likelihood. The logarithm of the likelihood function is given as

$$\begin{aligned} \text{Log}\, l(v|\xi) &= log\left(\frac{1}{1+\xi t}\right) - \left(\frac{1}{\xi}+1\right) log\left(1+\frac{\xi v}{1+\xi v}\right) \\ &= \frac{1}{\xi} log(1+\xi t) - \left(\frac{1}{\xi}+1\right) log(1+\xi t+\xi v). \end{aligned} \tag{12}$$

The prior assumed in this study is Jeffrey's prior given by the following equation

$$J(\xi) \propto \sqrt{|I(\xi)|} \tag{13}$$

where $I(\xi)$ is Fisher's information matrix given by

$$|I(\xi)| = E\left\{-\frac{d^2 log f(V|\xi)}{d\xi}\right\}. \tag{14}$$

$$\frac{dLogl(v|\xi)}{d\xi} = \frac{t}{\xi(1+\xi t)} - \frac{log(1+\xi t)}{\xi^2} - \frac{(1+\xi)(t+v)}{\xi(1+\xi t+\xi v)} + \frac{log(1+\xi t+\xi v)}{\xi^2} \tag{15}$$

and

$$\begin{aligned} &\frac{d^2 Logl(v|\xi)}{d\xi} = -\frac{t(1+2t\xi)}{(\xi+\xi^2 t)^2} + \frac{2log(1+\xi t)}{\xi^3} - \frac{t}{\xi^2(1+\xi t)} - \\ &\frac{(\xi+\xi^2 t+\xi^2 v)(t+v)-(1+\xi)(t+v)(1+2t\xi+2v\xi)}{(\xi+\xi^2 t+\xi^2 v)^2} - \frac{2log(1+\xi t+\xi v)}{\xi^3} + \frac{t+v}{\xi^2(1+\xi t+\xi v)}. \end{aligned} \tag{16}$$

The Jeffreys prior is therefore derived as follows:

$$J(\xi) \propto \sqrt{E\left(\frac{-d^2 logL(v|\xi)}{d\xi}\right)} =$$

$$\sqrt{\frac{t(1+2t\xi)}{\xi^2(1+\xi t)^2} + \frac{t}{\xi^2(1+\xi t)} - \frac{t(\xi+3)}{\xi^2(1+\xi)(1+\xi t)} - \frac{(\xi+3)}{\xi^2(1+\xi)} + \frac{2}{\xi^2} + \frac{t(1+\xi)}{\xi^2(1+2\xi)(1+\xi t)^2} + \frac{1}{\xi^2(1+2\xi)(1+\xi t)}} \quad (17)$$

The derivation of equation (17) is shown in more detail in Appendix A.1.

The posterior predictive survival distribution (11) can now be approximated as follows

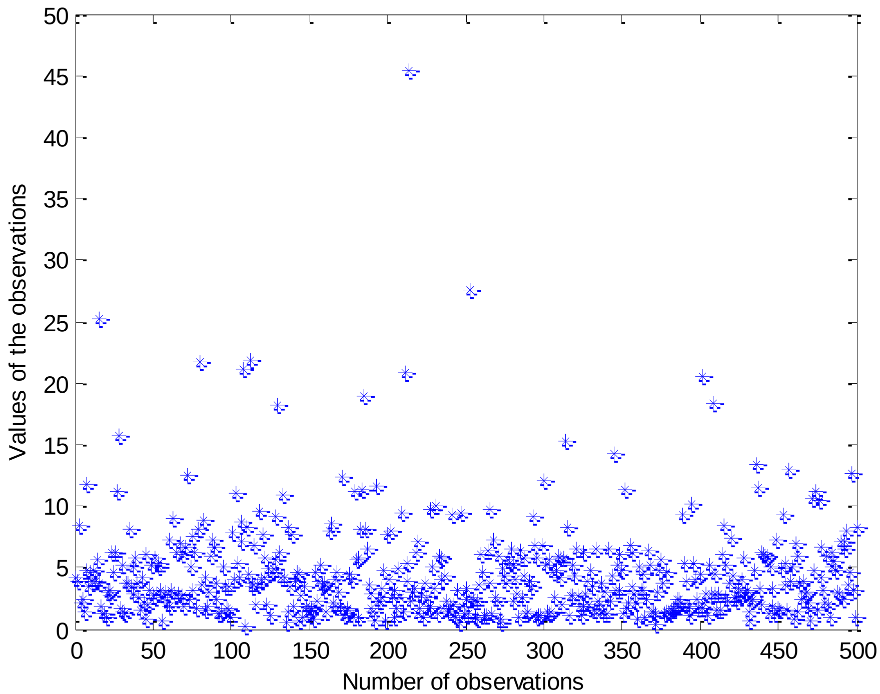
$$P(V_0 > v_0|v,t) \cong E_{\xi|v}\left(1 + \frac{\xi(v_0+t)}{1+\xi t}\right)^{-\frac{1}{\xi}} \quad (18)$$

for different values of ξ simulated from the posterior distribution. Equation (18) can now be used to calculate tail probabilities for a specific value of v_0. This process is illustrated in the next section through a practical application.

5. PRACTICAL APPLICATION THROUGH SIMULATION

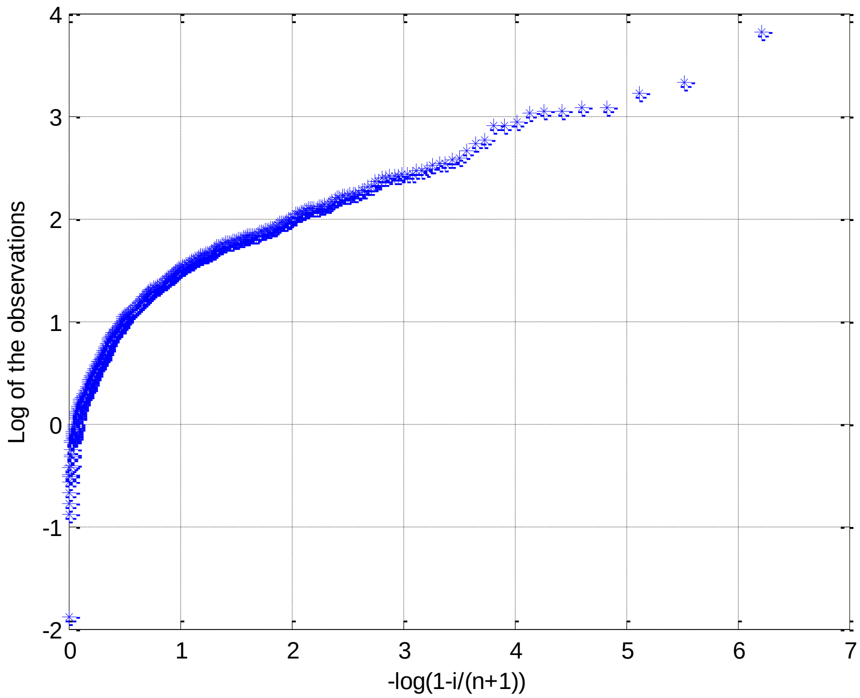
A data set of 500 values is simulated from a $\mathrm{GBG}(3,0,1,0.2)$. The following figure shows the 500 simulate v values.

Fig. 4 **500 simulated *v* values from a** $\mathbf{GBG(3, 0, 1, 0.2)}$



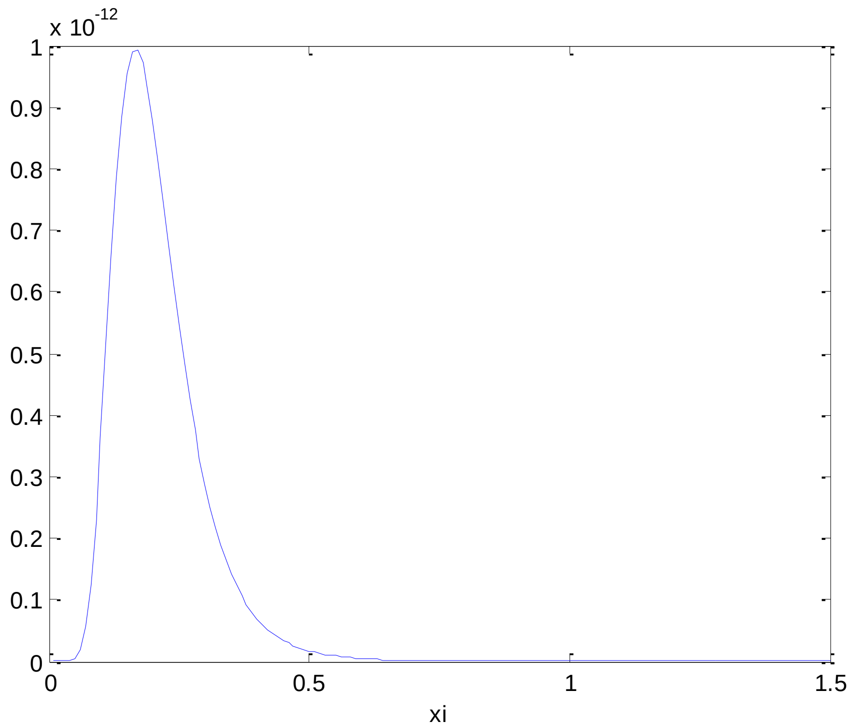
Next the Generalized Pareto quantile plot is considered to choose a threshold value (Beirlant *et al*., 2004). The Generalized Pareto quantile plot of the *v* values is given in the following figure. It seems as if the Generalized Pareto quantile plot follows a straight line at 2.9, therefore a threshold $t = \exp(2.9) = 18.1741$ is chosen.

Fig. 5 Generalized Pareto quantile plot on the v's



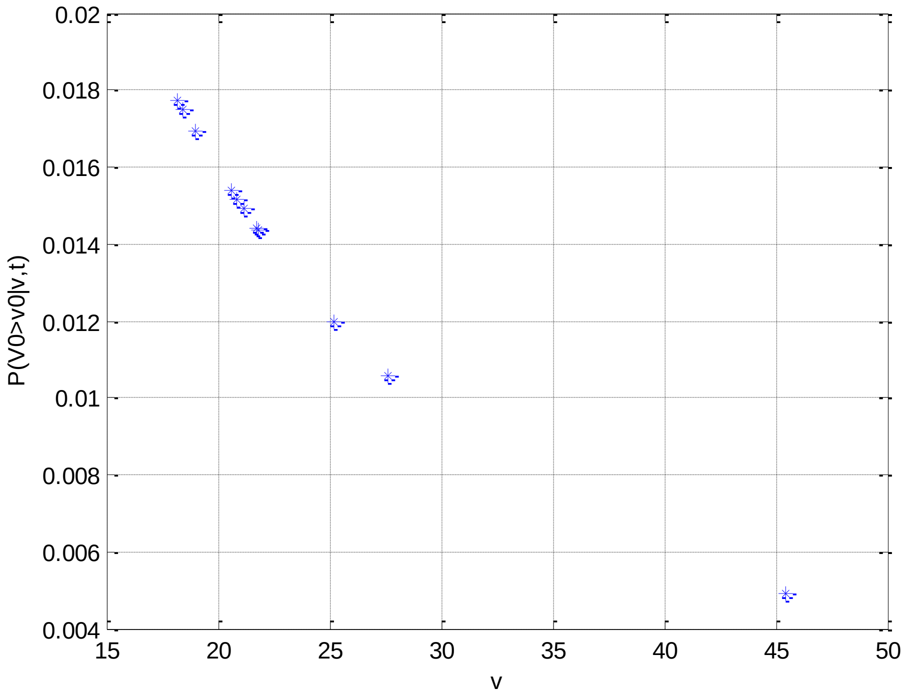
The v values above the threshold is now assumed to follow the approximated Generalized Pareto distribution which have one parameter ξ. ξ can be estimated as the value of ξ where the posterior distribution reaches a maximum. The following figure shows the posterior distribution for different values of $0 \geq \xi \leq 1.5$. The posterior distribution reaches a maximum at $\xi = 0.17$ and is thus our estimate for ξ. $\hat{\xi}$ is close to the actual simulated value of $\xi = 0.2$.

Fig. 6 **The posterior distribution for $0 \geq \xi \leq 1.5$**



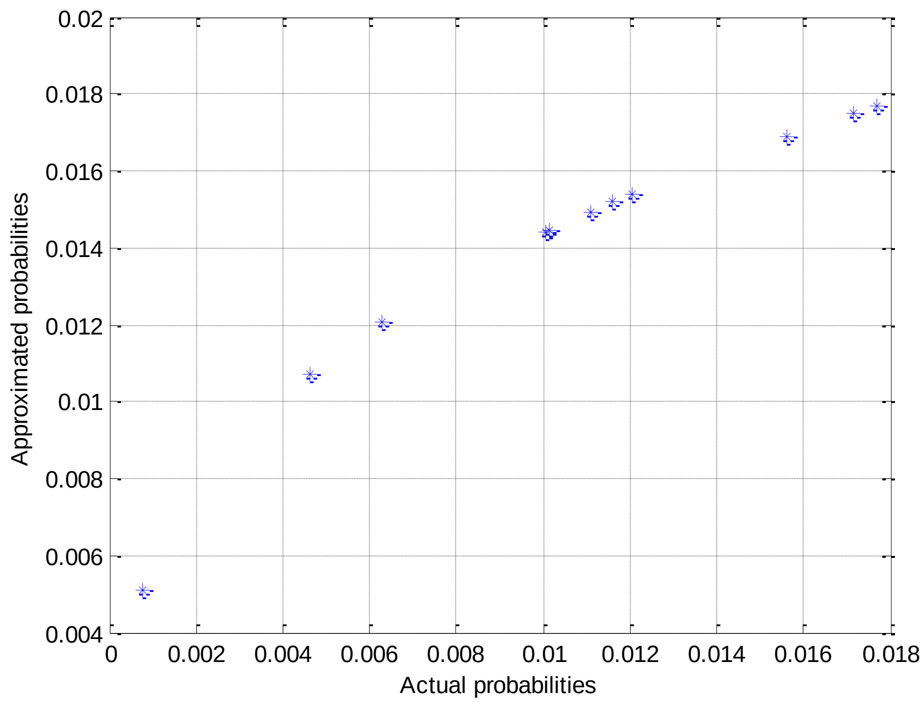
To see how good the approximated Generalized Pareto distribution fits the v data above the threshold, the tail probabilities for the v values above the threshold are calculated using equation (18), for different ξ values simulated from the posterior distribution. The following figure shows the posterior predictive survival function.

Fig.7 The posterior predictive survival function for values of *v* above the threshold



These tail probabilities are then compared to the tail probabilities of the actual GBG distribution. The following figure shows the actual tail probabilities plotted against the posterior predictive tail probabilities calculated for values of *v* above the threshold.

Fig. 8 **Actual tail probabilities plotted against the posterior predictive tail probabilities**

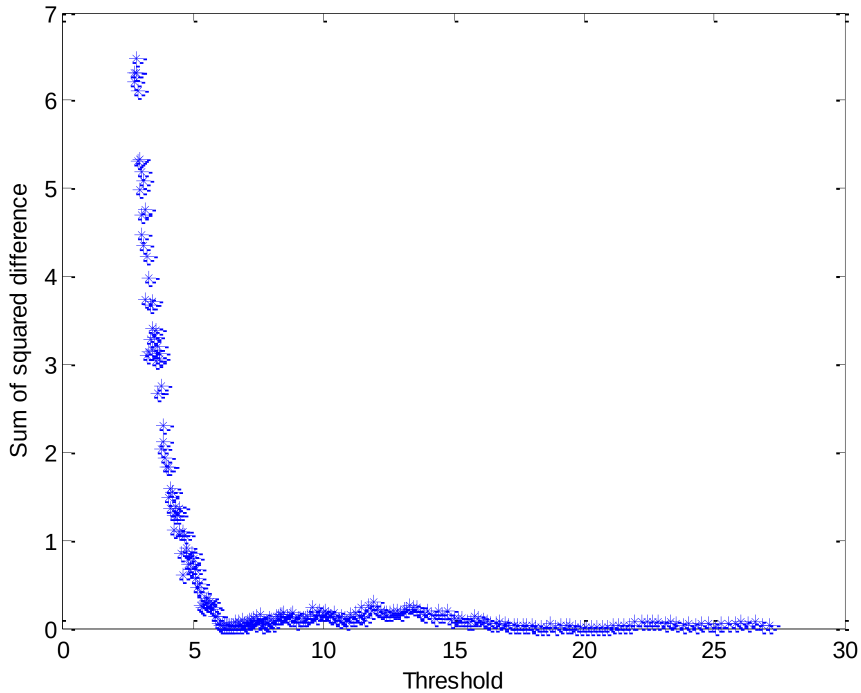


From Figure 8 it seems as if the values of the two tail probabilities are close. An indication of a good fit is when the tail probabilities lie close to the 45^0 line.

The sum of squared differences between the two probabilities is calculated as 0.0002 which is close to zero. The correlation between the two probabilities is calculated as a high value of 0.9578 also indicating that the approximated Generalized Pareto distribution is a good fit for the *v* values above the threshold.

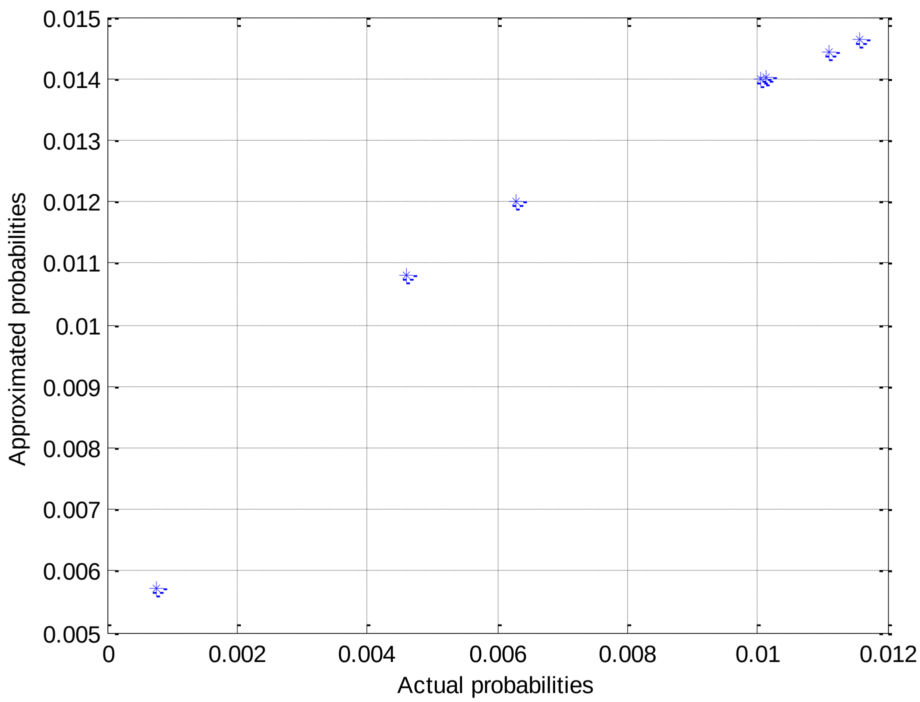
An ongoing question in extreme value theory is where to choose a reasonable and optimum threshold? Various studies have been done on how to select a threshold, see for example Beirlant *et al*., 2004 and Guillou and Hall, 2001. A recent paper by Peng, 2009, suggested an alternative method, to Guillou and Hall, 2001, for estimating a tail index. This alternative method of Peng, 2009 can also be extended to estimate the extreme value index and tail dependence function. In this study we consider selecting a threshold by minimizing the sum of squared differences between the two tail probabilities, $\sum_{i=1}^{n}(x_i - y_i)^2$, where x_i denotes the posterior predictive tail probabilities and y_i the actual tail probabilities from the GBG. For an appropriate threshold the sum of squared difference between the two tail probabilities should be small, close to zero. It can be shown that for different values of the threshold, the sum of squared difference between the two tail probabilities differs, as shown in Figure 9. Therefore, one can conclude that the threshold value that gives us the smallest sum of squared difference between the two tail probabilities will be the best threshold to use.

Fig. 9 The sum of squared difference for different values of the threshold



For this simulation study the minimum sum of squared difference, equal to 0.000017, was obtained at a threshold value of t = 20.8468. The estimated parameter value for ξ at this threshold is 0.14 which is rather close to the actual simulated parameter value of $\xi = 0.2$. The following figure shows the actual tail probabilities plotted against the approximated tail probabilities at this threshold value.

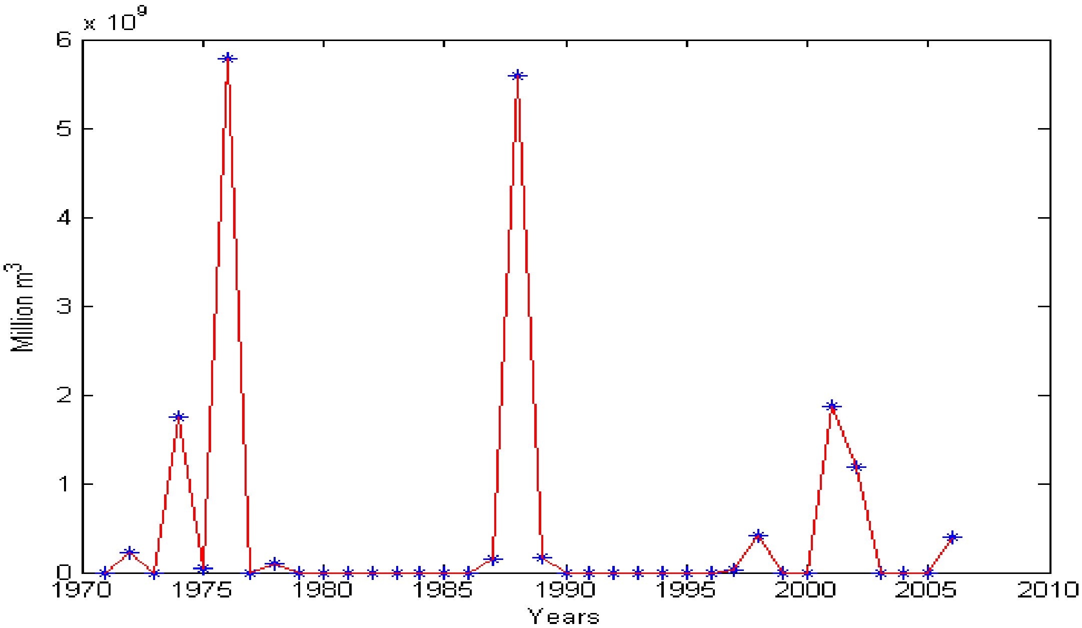
Fig. 10 Actual tail probabilities plotted against the posterior predictive tail probabilities



6. CHOOSING AN OPTIMUM THRESHOLD IN A REAL DATA SET

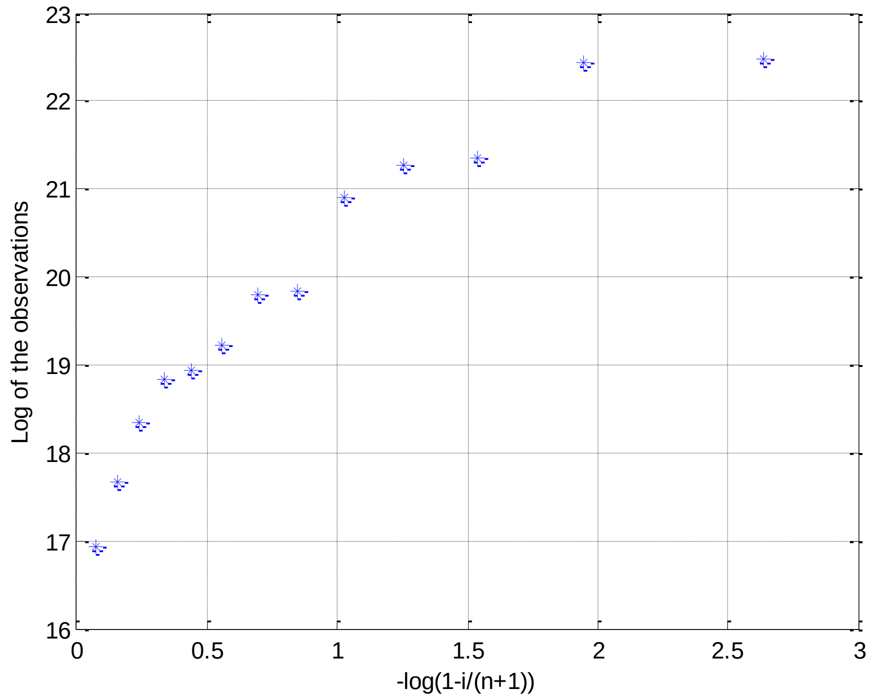
The data considered here is the total annual water spillage at the Gariep Dam during 1971 to 2006. The Gariep Dam is the largest reservoir in South Africa and lies in the upper Orange River. At full supply it stores 5943 million cubic meters of water. ESKOM, the main supplier of electricity in South Africa, has a hydro power station at the dam wall consisting of four turbines, each turbine can let through 162 cubic meters per second. If all 4 turbines are operating, the total release of water through the turbines is 648 m^3/s. Spillage over the wall will occur if the dam is 100% full with all 4 turbines running and the inflow into the dam exceeds 648 m^3/s. The total loss observed at Gariep due to spillage during 1971 to 2006 is $1.7693\text{x}10^{10}$ million cubic meters and in terms of South African Rand it was calculated as R76, 950, 708 which is a major loss. It is however important to note that out of the 36 years, 23 appeared without losses. Figure 11 shows the spillage during this period.

Fig. 11 Spillage at Gariep Dam during 1971-2006



It is shown in the article by Verster and De Waal (2009) that the water spillage data set can be considered to be Generalized Burr-Gamma (GBG) distributed. Verster and De Waal (2009) shows that the two parameters, μ and σ, is estimated through the method of moments when only considering the values Y below the threshold. The parameters k and ξ are estimated by calculating the Kolmogorov Smirnov measure, $KS = \max |F_n - F|$, (Conover, 1980) where F_n denotes the empirical cdf and F the fitted cdf. Since $V_\xi \sim GAM(k, 1)$ the Kolmogorov Smirnov measure calculates the maximum absolute difference between the empirical Gamma function and the cumulative Gamma function for different values of k and ξ. The k and ξ values that gives the minimum value of the different maximum Kolmogorov Smirnov measure values will be the estimates of k and ξ respectively (Verster and De Waal). Thus, the threshold value plays an important role in estimating the parameters. For different threshold values the parameter values can be estimated and the set of estimated parameter values that coincides with the optimum threshold can be found. The optimum threshold is again obtained when considering the minimum sum of squared difference between the two tail probabilities. The Generalized Pareto quantile plot of the water spillage data is given in the following figure.

Fig. 12 Generalized Pareto quantile plot



From Figure 12 the best threshold value for the spillage data was chosen among different threshold values t = exp(19.5) to t = exp(22.4). For every threshold values the sum of squared differences between the tail probabilities were calculated and the best threshold value was chosen as the threshold value that gives the minimum sum of squared difference. The threshold value of t = exp(20.5), on the water spillage data, and t = 26.0518, on the transformed v data, is chosen to be the best threshold. For this threshold value a minimum sum of squared difference of 0.0084 is obtained. The posterior distribution of the approximated Generalized Pareto distribution at this value of t is shown in the following figure, and an estimate of ξ, where the posterior is a maximum, is 0.201. At this value of the threshold the four parameters of the GBG by using the method of moments and Kolmogorov Smirnov measure respectively is calculated. Table 1 gives the estimated parameter values.

Fig. 13 The posterior distribution for $0 \geq \xi \leq 5$

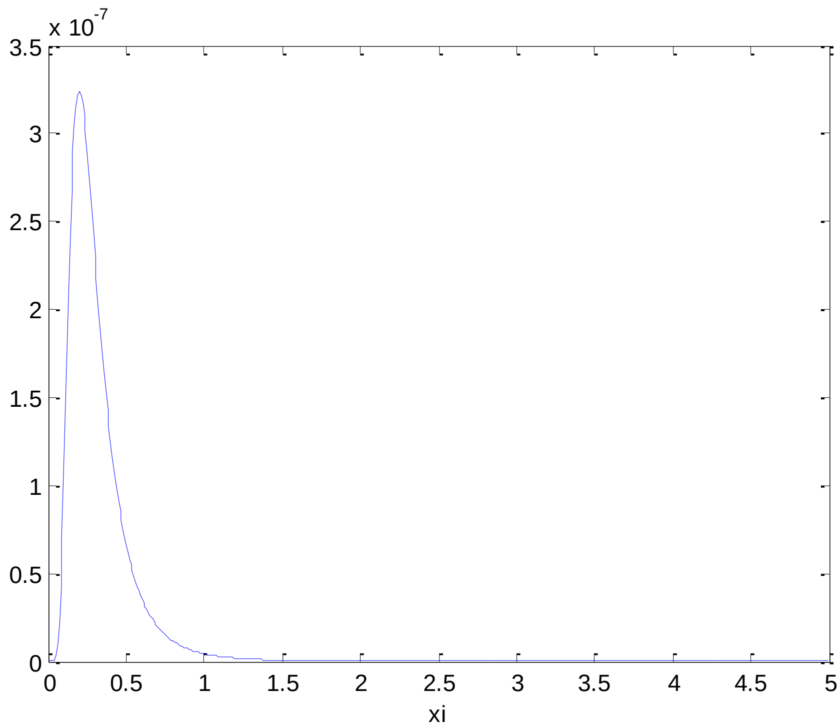
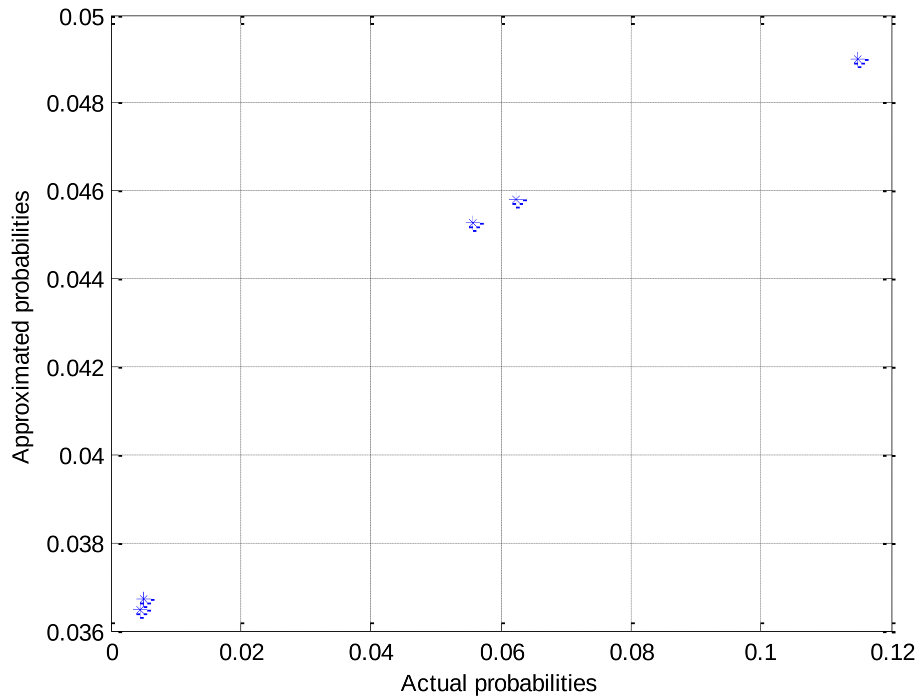


Table 1 The estimated parameter values of the GBG at t = exp(20.8)

$\hat{\mu}$	$\hat{\sigma}$	$\hat{k}$	$\hat{\xi}$
-18.6978	1.0241	17.5	0.02

The Figure 14 shows the GBG tail probabilities, calculated with the parameters in Table 1, plotted against the approximated tail probabilities of the approximated Generalized Pareto-type distribution when considering the threshold t = exp(20.5).

Fig. 14 Actual tail probabilities plotted against the posterior predictive tail probabilities



7. PREDICTING FUTURE OBSERVATONS

Future values can now be predicted by using the posterior predictive survival distribution of the approximated Generalized Pareto-type distribution given in equation (18). For the simulated data set, discussed in Section 5, we can for example predict the probability of obtaining a large value such as 98.39 namely $P(X \geq 98.39|x) = 0.0061$. The actual probability form the GBG distribution is $P(X \geq 98.39) = 0.0007$.

Another approach which is common in extreme values theory is to predict large quantiles. The p^{th} quantile is defined as the value x_p for which $P(X \leq x_p) = p$. For the simulated data the $(1-0.0061)^{th}$ quantile is calculated as 89.2741. The p^{th} quantile can now be predicted by using the predictive quantile distribution given in the following equation

$$Q_{Pred}(p) = \int Q_p(\xi)\, \pi(\xi|v,t) d\xi \qquad (19)$$

where $\pi(\xi|v,t)$ is the posterior distribution of the approximated Generalized Pareto-type of distribution and $Q_p(\xi)$ is the quantile function of the approximated Generalized Pareto-type of distribution given in the following equation

$$Q_p(\xi) = \left\{(1-p)^{-\xi} - 1\right\}\frac{(1+\xi t)}{\xi} - t. \tag{20}$$

Equation (19) can be approximated as

$$Q_{Pred}(p) \approx E_{\xi|p}\left(\left\{(1-p)^{-\xi} - 1\right\}\frac{(1+\xi t)}{\xi} - t\right). \tag{21}$$

$Q_{Pred}(1-0.0061)$ is predicted by using equation (21), where different values of ξ is simulated from the posterior distribution, as $Q_{Pred}(1-0.0061) = 96.4951$.

8. CONCLUSION

This study shows that the tail of a GBG distribution can be approximated with an approximated Generalized Pareto-type of distribution which is more convenient to work with because it has only one parameter ξ.

Further, the problem about choosing an optimum threshold is addressed here by considering the minimum sum of squared difference between the tail probabilities of the GBG and the tail probabilities of the approximated Generalized Pareto-type of distribution. This is a fairly easy and fast way to choose an optimum threshold.

An interesting question arising is: What is the difference between the Generalized Pareto-type and Generalized Pareto in approximating high quantiles? Can the Generalized Pareto-type be applied in general instead of the Generalized Pareto?

9. APPENDIX

A.1

Deriving Jeffreys prior, $J(\xi) \propto \sqrt{E\left(\frac{-d^2 logL(v|\xi)}{d\xi}\right)}$.

$$E\left(\frac{-d^2 logL(v|\xi)}{d\xi}\right)$$
$$= \frac{t(1+2t\xi)}{(\xi+\xi^2 t)^2} - \frac{2log(1+\xi t)}{\xi^3} + \frac{t}{\xi^2(1+\xi t)}$$
$$+ E\left\{\frac{(\xi+\xi^2 t+\xi^2 v)(t+v)-(1+\xi)(t+v)(1+2t\xi+2v\xi)}{(\xi+\xi^2 t+\xi^2 v)^2}\right.$$
$$\left.+\frac{2log(1+\xi t+\xi v)}{\xi^3} - \frac{t+v}{\xi^2(1+\xi t+\xi v)}\right\}$$

Let $A = \frac{t(1+2t\xi)}{(\xi+\xi^2 t)^2} - \frac{2log(1+\xi t)}{\xi^3} + \frac{t}{\xi^2(1+\xi t)}$ and $B = E\left\{\frac{(\xi+\xi^2 t+\xi^2 v)(t+v)-(1+\xi)(t+v)(1+2t\xi+2v\xi)}{(\xi+\xi^2 t+\xi^2 v)^2} + \frac{2log(1+\xi t+\xi v)}{\xi^3} - \frac{t+v}{\xi^2(1+\xi t+\xi v)}\right\}$.

First B is simplified as follows:

$$B = E\left\{-\frac{(t+v)}{(\xi+\xi^2 t+\xi^2 v)}\left(\frac{\xi+3}{\xi}\right) + \frac{(t+v)(1+\xi)}{(\xi+\xi^2 t+\xi^2 v)} + \frac{2log(1+\xi t+\xi v)}{\xi^3}\right\}$$

$$= \frac{1}{1+\xi t}\left\{\frac{-(\xi+3)}{\xi^2(1+\xi t)^{-\frac{1}{\xi}-1}}\int_0^\infty (t+v)(1+\xi t+\xi v)^{-\frac{1}{\xi}-2}dv\right.$$
$$+\frac{2}{\xi^3(1+\xi t)^{-\frac{1}{\xi}-1}}\int_0^\infty log(1+\xi t+\xi v)(1+\xi t+\xi v)^{-\frac{1}{\xi}-1}dv$$
$$\left.+\frac{(1+\xi)}{\xi^2(1+\xi t)^{-\frac{1}{\xi}-1}}\int_0^\infty (t+v)(1+\xi t+\xi v)^{-\frac{1}{\xi}-3}dv\right\}$$

Let C = $\int_0^\infty (t+v)(1+\xi t+\xi v)^{-\frac{1}{\xi}-2}dv$, D = $\int_0^\infty log(1+\xi t+\xi v)(1+\xi t+\xi v)^{-\frac{1}{\xi}-1}dv$ and

E = $\int_0^\infty (t+v)(1+\xi t+\xi v)^{-\frac{1}{\xi}-3}dv$. Integration by part is now considered to solve the intergrals.

C = $\left[\frac{-(t+v)}{(1+\xi)}(1+\xi t+\xi v)^{\frac{-1}{\xi}-1}\right]_0^\infty - \left[\frac{1}{(1+\xi)}(1+\xi t+\xi v)^{\frac{-1}{\xi}}\right]_0^\infty$

$$= \left[0+\frac{t}{1+\xi}(1+\xi t)^{\frac{-1}{\xi}-1}\right] - \left[0-\frac{1}{1+\xi}(1+\xi t)^{\frac{-1}{\xi}}\right]$$

$$= \frac{1}{1+\xi}(1+\xi t)^{\frac{-1}{\xi}}\left(\frac{t}{(1+\xi t)}+1\right)$$

D = $\left[-log(1+\xi t+\xi v)(1+\xi t+\xi v)^{\frac{-1}{\xi}}\right]_0^\infty - \left[\xi(1+\xi t+\xi v)^{\frac{-1}{\xi}}\right]_0^\infty$

$$= (1+\xi t)^{\frac{-1}{\xi}}[log(1+\xi t)+\xi]$$

E = $\left[\frac{-(t+v)}{(1+2\xi)}(1+\xi t+\xi v)^{\frac{-1}{\xi}-2}\right]_0^\infty - \left[\frac{(1+t\xi+\xi v)^{\frac{-1}{\xi}-1}}{(1+2\xi)(1+\xi)}\right]_0^\infty$

$= \frac{(1+\xi t)^{\frac{-1}{\xi}-1}}{(1+2\xi)}[t(1+\xi t)^{-1}+(1+\xi)^{-1}].$

Therefore

B = $\frac{-t(\xi+3)}{\xi^2(1+\xi)(1+\xi t)} - \frac{(\xi+3)}{\xi^2(1+\xi)} + \frac{2[log(1+\xi t)]}{\xi^3} + \frac{2}{\xi^2} + \frac{t(1+\xi)}{\xi^2(1+2\xi)(1+\xi t)^2} + \frac{1}{\xi^2(1+2\xi)(1+\xi t)}$

and

$E\left(-\frac{d^2 logL(v|\xi)}{d\xi}\right) = \frac{t(1+2t\xi)}{\xi^2(1+\xi t)^2} + \frac{t}{\xi^2(1+\xi t)} - \frac{t(\xi+3)}{\xi^2(1+\xi)(1+\xi t)} - \frac{(\xi+3)}{\xi^2(1+\xi)} + \frac{2}{\xi^2} + \frac{t(1+\xi)}{\xi^2(1+2\xi)(1+\xi t)^2} + \frac{1}{\xi^2(1+2\xi)(1+\xi t)}.$

therefore Jeffreys prior is $J(\xi) \propto$

$$\sqrt{\frac{t(1+2t\xi)}{\xi^2(1+\xi t)^2}+\frac{t}{\xi^2(1+\xi t)}-\frac{t(\xi+3)}{\xi^2(1+\xi)(1+\xi t)}-\frac{(\xi+3)}{\xi^2(1+\xi)}+\frac{2}{\xi^2}+\frac{t(1+\xi)}{\xi^2(1+2\xi)(1+\xi t)^2}+\frac{1}{\xi^2(1+2\xi)(1+\xi t)}}\,.$$

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