A Bayesian approach for comparing the relative returns of securities using the reciprocal of the coefficient of variation

By

Delson Chikobvu

Professor AJ van der Merwe

Dr Jesse (JA) De Beer

SUMMARY

In this scription, Bayesian statistics is employed to simulate and estimate the reciprocal of the coefficient of variation $\theta = \frac{\mu}{\sigma}$. The reciprocal of the coefficient of variation is used to compare the returns of three commodities, namely gold, platinum and oil. Results indicate that oil has a higher probability to outperform the other two commodities but there are no significant differences in the reciprocals of the coefficients of variation among the three commodities according to the credibility intervals (Bayesian confidence intervals) over the period from January 1980 to April 2007.

Bayesian inference has a number of advantages. A full Bayesian analysis provides a natural way of taking into account all sources of uncertainty in the estimation of the parameters. Uncertainty about the true value of the reciprocal of the coefficient of variation is incorporated into the analysis through the choice of a vague prior distribution.

The Bayesian simulation procedure employs the posterior distribution in doing the simulations. The procedure can be useful in solving the portfolio selection problem. Results show that the Bayesian simulation approach is just as good if not better than the standard classical statistical approach in assessing the performance of an investment. The added advantage of the Bayesian approach is that, from the posterior distribution of the reciprocal of the coefficient of variation, we are in a position to obtain quantiles, credible regions and perform other inferential tasks.

KEY WORDS: Bayesian analysis, Coefficient of variation, Moments, Monte Carlo simulation, Non-informative prior, Pearson's curve, Portfolio selection, Posterior distribution.

TABLE OF CONTENTS

1. Introduction

Portfolio selection is an important part of investment. It is often of interest to compare relative returns of three or more different investments such as gold, platinum and oil shares in order to make a choice among the three. Gold is often advertised as an inflation hedge, so many investors think that owning the metal provides constant insulation from the ravages of inflation (Zigler, 2009). But, what is a best inflation hedge; gold, platinum or oil?

The standard deviation can be used to compare risk among investments that have the **same** expected rate of return. The standard deviation is an *absolute* measure of dispersion of returns.

Another way to compare risk/return is to use the coefficient of variation. Suppose an investment has a return r. The return r is often assumed to be normally distributed with mean μ and variance σ^2. The coefficient of variation is defined as $CV = \frac{\sigma}{\mu}$ where σ is the standard deviation of return and μ is the expected rate of return. The coefficient of variation is used to compare the *relative* variability of two or more investments if there are major differences in the expected rates of return. The coefficient of variation indicates risk per unit of expected return. A larger value of the coefficient indicates greater risk (dispersion) relative to the mean rate of return (Reilly and Brown, 2003).

The reciprocal the coefficient of variation is defined as $\theta = \frac{\mu}{\sigma}$. It indicates expected return per unit of risk. A larger value of the coefficient indicates greater return relative to the risk measure.

In this paper a Bayesian simulation approach and Pearson curve approximations are used to asses the relative returns of three investments when there are major differences in the values of the risk measure (standard deviation). The reciprocal of the coefficient of variation is used to select among the three investments. This paper also investigates an overall testing procedure for testing the equality of $N = 3$ investments return (gold, platinum and oil) with differing values of risks. The analysis will be done from a Bayesian simulation point of view. The problem naturally occurs when comparing three or more investments.

To simplify matters we assume that an investor would like to compare three investments and determine which one will give the better performance based on relative return. The investor obtains data from each of the investments and estimates the coefficient of variation or it’s reciprocal. The objective of this problem then consists of identifying the investment that has the highest performance based on the observed estimates of the reciprocal of the coefficient of variation.

The following hypothesis is specifically of interest and would be considered:

$$H_o : \frac{\mu_1}{\sigma_1} = \frac{\mu_2}{\sigma_2} = \frac{\mu_3}{\sigma_3}$$

$H_1 : \frac{\mu_i}{\sigma_i} \neq \frac{\mu_k}{\sigma_k}$ for at least at one value of $i \neq k \ (i = 1,2,3 \ k = 1,2,3)$.

μ_i and σ_i are the mean and the standard deviation respectively, of each of the investments.

2. Literature review

In Bayesian analysis, we assume that we have prior knowledge or information or opinion about parameters of a statistical distribution and very often in practice we do. We then attach a distribution to this belief. Parameters do not really have a distribution, parameters are constants, and so a prior distribution is a way of expressing our belief or opinion on our parameters. A posterior distribution is the belief distribution of the parameters after the outcomes of experiments (data) have been observed. There is now an updated belief distribution in light of the information from the data (Hoshino, 2008).

To explain it in more detail: The information contained in the prior is combined with the likelihood function (distribution of the data) to obtain the posterior distribution of the parameters. Inferences about the unknown parameters are based on the posterior distribution. If the form of the posterior distribution is complicated, Monte Carlo simulation procedures or numerical methods like Pearson curves approximations or Cornish–Fisher expansions can be used to solve different complex problems such as hypothesis testing, credibility intervals (Bayesian confidence intervals) and ranking and selection.

Let r_{ij} be return for period j from a random sample of size n_i observations from the $i^{th}(i=1,2,3)$ investment, and we have

$\bar{r}_i = \frac{\sum_{j=1}^{n_i} r_{ij}}{n_i}$ and $s_i = \sqrt{\frac{\sum_{i=1}^{n_i}(r_{ij}-\bar{r}_i)^2}{n_i-1}}$ as estimates of μ_i and σ_i respectively.

It is assumed that there is no restriction on sample sizes drawn from the $N=3$ investments. A random sample in these cases is assumed to imply that the random variables (namely gold, platinum and oil) observed from each of the investment are mutually independent and identically distributed. The classical estimate of the reciprocal of the coefficient of variation θ is

$$\hat{\theta}_i = \frac{\bar{r}_i}{s_i}$$

2.1 Bayesian model

We assume that $r_{ij}(i=1,2,3\ j=1,2\cdots n_i)$ are independently and identically normally distributed with mean μ_i and variance $\sigma_i^{\,2}$. Since both μ_i and $\sigma_i^{\,2}$ are unknown and only 'vague' information is available, the conventional non informative prior:

$$\pi(\mu_i,\sigma_i^{\,2}) \propto \sigma_i^{\,-2} \tag{2. 1}$$

will be specified for them.

Equation (2.1) is however the reference prior and it is also a probability-matching prior. It is also called the independence Jeffreys' prior because it is obtained from the Fisher information matrix for $\sigma_i^{\,2}$ only, i.e. treating the location parameter μ_i separately from the variance component $\sigma_i^{\,2}$. According to Box and Tiao (1973) it is usually appropriate to take location parameters to be distributed independently of scale parameters.

Using (2.1), it is well known (see for example Zellner, 1971) and it is proved in the Appendix that the conditional posterior density of μ_i is normal:

$$\mu_i \mid \sigma_i^{\,2}, \underline{r_i} \sim N\left(\bar{r}_i, \frac{\sigma_i^{\,2}}{n_i}\right) \tag{2. 2}$$

and the posterior density for the variance component $\sigma_i^{\,2}$, is given by

$$p(\sigma_i^{\,2} \mid \underline{r_{i.}}) = C(\sigma_i^{\,2})^{-\frac{1}{2}(n_i-1)-1} \exp\left\{-\tfrac{1}{2}(n_i-1)s_i^{\,2}/\sigma_i^{\,2}\right\} \quad \sigma_i^{\,2} > 0 \tag{2. 3}$$

$$= IG(\sigma_i^2 \mid \underline{r_{i.}})$$

an inverted gamma density, where $\underline{r_{i.}} = [r_{i1}, r_{i2}\ldots, r_{in_i}]'$, , and the normalizing constant is

$$C = \left\{ \frac{(n_i - 1)s_i^{\,2}}{2} \right\}^{\frac{1}{2}(n_i - 1)} \frac{1}{\Gamma\left(\frac{n_i - 1}{2}\right)}.$$

$$IG(x \mid \beta_i, \alpha_i) = \frac{\alpha^{\beta_i}}{\Gamma(\beta_i)} x^{-\beta_i - 1} \exp(-\alpha_i / x) \qquad (2.4)$$

i.e. the inverted gamma density with positive parameters $\beta_i = \frac{1}{2}(n_i - 1)s_i^{\,2}$ and $\alpha_i = \frac{1}{2}(n_i - 1)$.

From (2.3) it follows that $\frac{(n_i - 1)s_i^{\,2}}{\sigma_i^{\,2}} \sim \chi^2_{(n_i - 1)}$. (2.5)

From classical statistics (if s_i^2 is considered to be random) it is well known that $\frac{(n_i - 1)s_i^{\,2}}{\sigma_i^{\,2}}$ is also distributed chi-square with $(n_i - 1)$ degrees of freedom. This agreement between classical and Bayesian statistics is only true if the prior $\pi(\mu_i, \sigma_i^{\,2}) \propto \sigma_i^{\,-2}$ is used. If some other prior distributions are used, for example $\pi(\mu_i, \sigma_i^{\,2}) \propto \sigma_i^{\,-3}$ or $\pi(\mu_i, \sigma_i^{\,2}) \propto \text{constant}$ then the posterior distribution of $\frac{(n_i - 1)s_i^{\,2}}{\sigma_i^{\,2}}$ will still be a chi-square distribution but the degrees of will be different.

3. The posterior distribution of $\theta = \frac{\mu}{\sigma}$

As mentioned from a Bayesian point of view posterior distributions are of importance, in evaluating θ. One of the aims of this note is therefore to derive the exact posterior distribution of $\theta = \frac{\mu}{\sigma}$ and also approximated posterior distributions of $d_{ik} = \theta_i - \theta_k$ $(i = 1, 2, 3 \quad k = 1, 2, 3$ and $i \neq k)$.

The following theorem can be proved.

Theorem 3.1

The posterior distribution of $\theta = \frac{\mu}{\sigma}$ is given by

$$p\left(\theta \middle| \hat{\theta}\right) = \frac{\sqrt{n}\exp\left(-\frac{n\theta^2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{2\pi}} \sum_{l=0}^{\infty} \left(\frac{n\theta\hat{\theta}}{\sqrt{\nu}}\right)^l \frac{\Gamma\left(\frac{\nu+l}{2}\right)2^{\frac{1}{2}l}}{l!\left(1+\frac{n}{\nu}\hat{\theta}^2\right)^{\frac{1}{2}(\nu+l)}} \quad -\infty < \theta < \infty \tag{3.1}$$

where $\nu = n-1$, $\hat{\theta} = \frac{\bar{y}}{s}$, $\bar{y} = \sum_{j=1}^{n} y_i$ and $s = \sqrt{\frac{\sum_{j=1}^{n}(y_j - \bar{y})^2}{n-1}}$.

<u>Proof</u>

The proof is given in the Appendix.

As far as we know, the derivation of the posterior distribution of $\theta = \frac{\mu}{\sigma}$ (equation 3.1) has never been done before.

It is easy to prove (see also Chen and Owen,1989) that the distribution of $\hat{\theta} = \frac{\bar{y}}{s}$ is given by

$$p\left(\hat{\theta} \middle| \theta\right) = \frac{\sqrt{n}\exp\left(-\frac{n\theta^2}{2}\right)}{\sqrt{\nu}\sqrt{2\pi}\Gamma\left(\frac{\nu}{2}\right)} \sum_{l=0}^{\infty} \left(\frac{n\hat{\theta}\theta}{\sqrt{\nu}}\right)^l \frac{\Gamma\left(\frac{\nu+l+1}{2}\right)2^{\frac{1}{2}l}}{l!\left(1+\frac{n}{\nu}\hat{\theta}^2\right)^{\frac{1}{2}(\nu+l+1)}} \quad -\infty < \theta < \infty \tag{3.2}$$

which is a non-central t-distribution with ν degrees of freedom and the non-centrality parameter $\delta = n\theta^2 = \frac{n\mu^2}{\sigma^2}$.

The two density functions (equations 3.1 and 3.2) look similar but are infact quite different. In equation (3.1) $\theta = \frac{\mu}{\sigma}$ is the random variable while in (3.2) $\hat{\theta} = \frac{\bar{y}}{s}$ is the random variable.

Although it is not the purpose of this note to study the coverage properties of the Bayesian confidence or to look at the frequential aspects of the Bayesian procedure, the long time (long run) properties of $E\left(\theta\middle|\hat{\theta}\right)$ (the expected value of the posterior distribution) will be of interest. Indeed some statisticians argue that frequency calculations are an important part of applied Bayesian statistics.

From equation (3.2) it follows that:

$$E\left(\hat{\theta}\middle|\theta\right)=\left(\frac{v}{2}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\theta \tag{3.3}$$

is not an unbiased estimate of θ. Infact $E\left(\hat{\theta}\middle|\theta\right)>\theta$

For $n=10$, $E\left(\hat{\theta}\middle|\theta\right)=1.0942\theta$ and

for $n=20$, $E\left(\hat{\theta}\middle|\theta\right)=1.0418\theta$

From equation (3.1) it can be shown that

$$E\left(\theta\middle|\hat{\theta}\right)=\left(\frac{v}{2}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\hat{\theta} \tag{3.3}$$

which underestimates $\hat{\theta}$

For $n=10$, $E\left(\theta\middle|\hat{\theta}\right)=0.9727\hat{\theta}$ and

for $n=20$, $E\left(\theta\middle|\hat{\theta}\right)=0.9869\hat{\theta}$

Therefore in the long run,

$$E_{\hat{\theta}}\left[E\left(\theta\middle|\hat{\theta}\right)\right]=\left(\frac{v}{2}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma^{2}\left(\frac{\nu}{2}\right)}\theta$$

Which is nearly unbiased and an important improvement on $E\left(\hat{\theta}\middle|\theta\right)$.

For $n=10$, $E_{\hat{\theta}}\left[E\left(\theta\middle|\hat{\theta}\right)\right]=\left(0.9727\right)\left(1.0942\right)\theta=1.0643\theta$

and for $n = 20$, $E_{\hat{\theta}}\left[E\left(\theta\middle|\hat{\theta}\right)\right] = (0.9869)(1.0418)\theta = 1.0282\theta$

This shows that the Bayesian procedure is just as good or even better than the classical procedure.

4. Methodology

Consider the three investments which are suspected to differ widely in return and risk (as measured by the standard deviation of returns). From (2.2) and (2.3) it follows that:

$$\mu_i \mid \underline{r_i}, \sigma_i^{\ 2} \sim N(\bar{r}_i, \frac{\sigma_i^{\ 2}}{n_i}) \text{ and } \frac{\nu_i s_i^{\ 2}}{\sigma_i^{\ 2}} \sim \chi^2_{\nu_i} \quad \text{for} \quad i = 1, 2, 3$$

where $\nu_i = (n_i - 1)$ and $\nu_i s_i^{\ 2} = \sum_{j=1}^{n_i} (r_{ij} - \bar{r}_i)^2$.

4.1 Bayesian significance testing of equality of the reciprocal of the coefficients of variation.

In this subsection, we show a method of estimating the p-value of the hypothesis that all the reciprocal of the coefficients of variation for all the commodities are equal.

$\mu_i \mid \sigma_i^{\ 2}, \underline{r_{i.}} \sim N\left(\bar{r}_{i.}, \frac{\sigma_i^{\ 2}}{n_i}\right)$ and therefore $\frac{\mu_i}{\sigma_i} \mid \underline{r_{i.}} \sim N\left(\frac{\bar{r}_i}{\sigma_i}, \frac{1}{n_i}\right)$.

$$\left[\begin{array}{c|c} \frac{\mu_1}{\sigma_1} & \\ \frac{\mu_2}{\sigma_2} & \sigma_1, \sigma_2, \sigma_3 \\ \frac{\mu_3}{\sigma_3} & \end{array}\right] = N\left\{\begin{bmatrix} \frac{\bar{r}_1}{\sigma_1} \\ \frac{\bar{r}_2}{\sigma_2} \\ \frac{\bar{r}_3}{\sigma_3} \end{bmatrix}, \begin{pmatrix} \frac{1}{n_1} & 0 & 0 \\ 0 & \frac{1}{n_2} & 0 \\ 0 & 0 & \frac{1}{n_3} \end{pmatrix}\right\}$$

Therefore for given σ_1, σ_2 and σ_3

$\underline{\theta} \sim N(\underline{\xi}, \Sigma)$

where

$$\underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} \frac{\mu_1}{\sigma_1} \\ \frac{\mu_2}{\sigma_2} \\ \frac{\mu_3}{\sigma_3} \end{bmatrix}, \quad \underline{\xi} = \begin{bmatrix} \frac{\bar{r}_1}{\sigma_1} \\ \frac{\bar{r}_2}{\sigma_2} \\ \frac{\bar{r}_3}{\sigma_3} \end{bmatrix} \text{ and } \Sigma = \begin{pmatrix} \frac{1}{n_1} & 0 & 0 \\ 0 & \frac{1}{n_2} & 0 \\ 0 & 0 & \frac{1}{n_3} \end{pmatrix}$$

The following null hypothesis:

$H_o : \theta_1 = \theta_2 = \theta_3$

can be split as:

$H_o : \theta_1 = \theta_2 = 0$

$\theta_1 = \theta_3 = 0$

to allow us to write the hypothesis in matrix form:

$H_o : C\,\underline{\theta} = 0$ where $C = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

Also for given σ_1, σ_2 and σ_3

$C\underline{\theta} \sim N\left(C\underline{\xi}, C\Sigma C'\right)$

Therefore we can now use the Chi-square test statistic

$T^2 = \left[C\underline{\theta} - C\underline{\xi}\right]'\left(C\Sigma C'\right)^{-1}\left[C\underline{\theta} - C\underline{\xi}\right] \sim \chi^2_{rank\,C}$ to test for significant differences (Hoshino, 2008).

If H_o is true, then

$T^2 = \left[C\underline{\xi}\right]'\left(C\Sigma C'\right)^{-1}\left[C\underline{\xi}\right]$ gives a χ^2_2 value.

σ_1, σ_2 and σ_3 are simulated from the posterior distribution. The p-value is not calculated analytically. Monte-carlo simulation is used to calculate the approximate p-value of the hypothesis (Hoshino, 2008).

4.2 Simulation of the reciprocal of the coefficient of variation.

The standard routines are used to simulate each of the the reciprocals of the coefficient of variation (Berger and Sun, 2008).

1. Simulation of $\sigma_i^{\,2}$ can be obtained in the following way:

   (a) Simulate a $\chi^2_{\nu_i}$ variate, as a sum of ν_i squared independent standard normal random variates.

   (b) Calculate $\sigma_i^{\,2*} = \dfrac{\nu_i s_i^{\,2}}{\chi^2_{\nu_i}}$ where (*) indicates a simulated value.

   (c) $\sigma_i^{\,*} = \sqrt{\dfrac{\nu_i s_i^{\,2}}{\chi^2_{\nu_i}}}$

2. By making use of the fact that $\mu_i \mid \underline{r_i}, \sigma_i^{\,2} \sim N(\bar{r}_i, \dfrac{\sigma_i^{\,2}}{n_i})$, where $\underline{r_{i.}}$ is data drawn from investment i, simulate μ_i and from the definition of θ_i, it follows that θ_i can be simulated as

$$\theta_i^* = \left(\frac{\mu_i^{\,*}}{\sigma_i^{\,*}}\right) = \left(\frac{\bar{r}_i + Z_i\sqrt{\dfrac{\sigma_i^{\,2*}}{n_i}}}{\sigma_i^{\,*}}\right) \quad \text{(*) indicates a simulated value.}$$

$$= \left(\bar{r}_i \frac{1}{\sigma_i^{\,*}} + Z_i\sqrt{\frac{1}{n_i}}\right) = \left(\bar{r}_i\sqrt{\frac{\chi^2_{\nu_i}}{\nu_i s_i^{\,2}}} + Z_i\sqrt{\frac{1}{n_i}}\right)$$

where $Z_i \sim N(0,1) \quad (i = 1,2,3)$.

4.3 Approximated distributions of the differences between the reciprocal of the coefficients of variation.

As mentioned in section 3, one of the aims of this note is to derive approximated posterior distributions for $d_{ik} = \theta_i - \theta_k \; (i = 1,2,3 \quad k = 1,2,3$ and $i \neq k)$ which will be used for pair-wise testing. Exact posterior distributions of the differences are difficult to derive but they can be approximated by

(I) Monte Carlo simulation
(II) Pearson curve approximations
(III) Cornish-Fisher expansions

4.3.1 Simulation of the differences between the reciprocal of the coefficients of variation.

Let $d_{ik} = \theta_i - \theta_k$ be the difference between the reciprocal of the coefficients of variation for any two of the investments.

$d_{ik} = \left(\frac{\mu_i}{\sigma_i}\right) - \left(\frac{\mu_k}{\sigma_k}\right)$ and d_{ik} can now be simulated as follows

$$d_{ik}^* = \left(\bar{r}_i \sqrt{\frac{\chi^2_{\nu_i}}{\nu_i s_i^2}} - \bar{r}_k \sqrt{\frac{\chi^2_{\nu_k}}{\nu_k s_k^2}} + Z_i \sqrt{\frac{1}{n_i}} - Z_k \sqrt{\frac{1}{n_k}} \right)$$

where Z_i and Z_k are independently normally distributed with mean 0 and variance 1. $\chi^2_{\nu_i}$ and $\chi^2_{\nu_k}$ are independent chi-square variables with ν_i and ν_k degrees of freedom respectively.

The null hypothesis $H_0 : \theta_i = \theta_k$ is rejected when zero is not included in the $100(1-\alpha)\%$ credible region of the posterior distribution of $d_{ik} = \theta_i - \theta_k$.

4.3.2 Plotting the distribution of the differences between the reciprocal of the coefficients of variation using Pearson curves.

To obtain Pearson curves we must first derive the mean, variance, third and fourth moments about the mean for d_{ik}. As far as we know, these moments have not been derived before.

The distribution of for instance d_{12} given $\chi^2_{\nu_1}, \chi^2_{\nu_2}$ is the normal distribution and is expressed as:

$$d_{12} \mid \chi^2_{\nu_1}, \chi^2_{\nu_2}, \underline{r_{i.}} \sim N\left(\bar{r}_1 \sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 s_1^{\,2}}} - \bar{r}_2 \sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2^{\,2}}}, \quad \left(\frac{1}{n_1} + \frac{1}{n_2} \right) \right)$$

The following theorem can now be stated.

Theorem 4.1

For given $\chi^2_{\nu_1}$ and $\chi^2_{\nu_2}$, denote the first four posterior moments about the origin for d_{12} by μ_1, μ'_2, μ'_3 and μ'_4 and the central moments by μ_2, μ_3 and μ_4, then:

$$\mu'_1 = \bar{r}_1 \sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 s_1^{\,2}}} - \bar{r}_2 \sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2^{\,2}}}$$

$$\mu'_2 = \left(\frac{1}{n_1} + \frac{1}{n_2} \right) + \left(\frac{(\bar{r}_1)^2 (\chi^2_{\nu_1})}{(\nu_1 s_1^{\,2})} - 2\bar{r}_1 \bar{r}_2 \sqrt{\frac{\chi^2_{\nu_1} \chi^2_{\nu_2}}{\nu_1 s_1^{\,2} \nu_2 s_2^{\,2}}} + \frac{(\bar{r}_2)^2 (\chi^2_{\nu_2})}{(\nu_2 s_2^{\,2})} \right)$$

$$\mu'_3 = 3\left(\frac{1}{n_1} + \frac{1}{n_2} \right)\left(\bar{r}_1 \sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 s_1^{\,2}}} - \bar{r}_2 \sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2^{\,2}}} \right) + \frac{\bar{r}_1^{\,3} (\chi^2_{\nu_1})^{\frac{3}{2}}}{(\nu_1 s_1^{\,2})^{\frac{3}{2}}}$$

$$-3 \frac{(\bar{r}_1)^2 (\chi^2_{\nu_1})}{(\nu_1 s_1^{\,2})} \frac{(\bar{r}_2)(\chi^2_{\nu_2})^{\frac{1}{2}}}{(\nu_2 s_2^{\,2})^{\frac{1}{2}}} + 3 \frac{(\bar{r}_1)(\chi^2_{\nu_1})^{\frac{1}{2}}}{(\nu_1 s_1^{\,2})^{\frac{1}{2}}} \frac{(\bar{r}_2)^2 (\chi^2_{\nu_2})}{(\nu_2 s_2^{\,2})} - \frac{(\bar{r}_2)^3 (\chi^2_{\nu_2})^{\frac{3}{2}}}{(\nu_2 s_2^{\,2})^{\frac{3}{2}}}$$

$$\mu'_4 = 3\left(\frac{1}{n_1}+\frac{1}{n_2}\right)^2 + 6\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left(\frac{(\bar{r}_1)^2(\chi^2_{\nu_1})}{(\nu_1 s_1{}^2)} - 2\bar{r}_1\bar{r}_2\sqrt{\frac{\chi^2_{\nu_1}\chi^2_{\nu_2}}{\nu_1 s_1{}^2 \nu_2 s_2{}^2}} + \frac{(\bar{r}_2)^2(\chi^2_{\nu_2})}{(\nu_2 s_2{}^2)}\right)$$

$$+\frac{(\bar{r}_1)^4(\chi^2_{\nu_1})^2}{(\nu_1 s_1{}^2)^2} + 4\frac{(\bar{r}_1)^3(\bar{r}_2)(\chi^2_{\nu_1})^{\frac{3}{2}}}{(\nu_1 s_1{}^2)^{\frac{3}{2}}}\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2{}^2}} + 6\frac{(\bar{r}_1)^2(\bar{r}_2)^2(\chi^2_{\nu_1})(\chi^2_{\nu_2})}{(\nu_1 s_1{}^2)(\nu_2 s_2{}^2)}$$

$$-4(\bar{r}_1)(\bar{r}_2)^3\frac{(\chi^2_{\nu_1})^{\frac{1}{2}}(\chi^2_{\nu_2})^{\frac{3}{2}}}{(\nu_1 s_1{}^2)^{\frac{1}{2}}(\nu_2 s_2{}^2)^{\frac{3}{2}}} + \frac{(\bar{r}_2)^4(\chi^2_{\nu_2})^2}{(\nu_2 s_2{}^2)^2}$$

<u>Proof</u>

The proof is given in Appendix.

The following theorem can now be stated.

Theorem 4.2

Denote the first four posterior moments about the origin for d_{12} (unconditional) by $m_1^{'}, m_2^{'}, m_3^{'} \quad and \quad m_4^{'}$ and also denote the (unconditional) variance, third and fourth central moments of the difference $d_{12} = \theta_1 - \theta_2$ by $m_2, m_3 \quad and \quad m_4$, then

$$m_1^{'} = E(d_{12} \mid \underline{r_{i.}}) = \sqrt{2}\left(\frac{\bar{r}_1}{\sqrt{\nu_1 s_1{}^2}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)} - \frac{\bar{r}_2}{\sqrt{\nu_2 s_2{}^2}}\frac{\Gamma(\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right) \text{ for } i = 1, 2$$

$$m_2 = Var(d_{12} \mid \underline{r_{i.}}) = \left(\frac{1}{n_1}+\frac{1}{n_2}\right) + \left(\frac{(\bar{r}_1)^2}{\nu_1 s_1{}^2}\left\{\nu_1 - \frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right\} + \frac{(\bar{r}_2)^2}{\nu_2 s_2{}^2}\left\{\nu_2 - \frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}\right)$$

$$m_3 = (\bar{r}_1)^3\left(\frac{2}{\nu_1 s_1{}^2}\right)^{\frac{3}{2}}\frac{\Gamma^3\left(\frac{\nu_1+1}{2}\right)}{\Gamma^3\left(\frac{\nu_1}{2}\right)}\left\{\frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)} - \frac{(2\nu_1-1)}{2}\right\} - (\bar{r}_2)^3\left(\frac{2}{\nu_2 s_2{}^2}\right)^{\frac{3}{2}}\frac{\Gamma^3\left(\frac{\nu_2+1}{2}\right)}{\Gamma^3\left(\frac{\nu_2}{2}\right)}\left\{\frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)} - \frac{(2\nu_2-1)}{2}\right\}$$

$$m_4 = 3\left(\frac{1}{n_1}+\frac{1}{n_2}\right)^2 + 6(\overline{r_1})^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\frac{1}{(\nu_1 s_1{}^2)}\left\{\nu_1 - \frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right\}$$

$$+6(\overline{r_2})^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\frac{1}{(\nu_2 s_2{}^2)}\left\{\nu_2 - \frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}$$

$$+\frac{3(\overline{r_1})^4}{(\nu_1 s_1{}^2)^2}\left\{\frac{\nu_1(\nu_1+2)}{3}+4\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\left[\frac{1}{3}(\nu_1-2)-\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right]\right\}$$

$$+\frac{3(\overline{r_2})^4}{(\nu_2 s_2{}^2)^2}\left\{\frac{\nu_2(\nu_2+2)}{3}+4\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\left[\frac{1}{3}(\nu_2-2)-\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right]\right\}$$

$$+\frac{6(\overline{r_1})^2(\overline{r_2})^2}{(\nu_1 s_1{}^2)(\nu_2 s_2{}^2)}\left\{\nu_1\nu_2+4\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}-\frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)\nu_2}{\Gamma^2\left(\frac{\nu_1}{2}\right)}-\frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)\nu_1}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}$$

<u>Proof</u>

The proof is given in Appendix.

If $n_1 = n_2 = n$ and $\nu = (n-1)$, then

$$E(d_{12} \mid \underline{r_{i.}}) = \sqrt{2}\,\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{\overline{r_1}}{\sqrt{\nu s_1{}^2}}-\frac{\overline{r_2}}{\sqrt{\nu s_2{}^2}}\right)$$

and

$$Var(d_{12} \mid \underline{r_{i.}}) = \frac{2}{n}+\left\{\nu-\frac{2\Gamma^2\left(\frac{\nu+1}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)}\right\}\left(\frac{(\overline{r_1})^2}{\nu s_1{}^2}+\frac{(\overline{r_2})^2}{\nu s_2{}^2}\right)$$

For details of how to determine the parameters of a Pearson curve, given the values of its moments, see for example Elderton (1953) or Elderton and Johnson (1969). The advantage of the Pearson curve approximation is that the formula of the density can be obtained. A type I Pearson curve can be used to approximate the posterior distribution of d_{12} for our data set.

The density of a type I curve is given by:

$$f(d_{12}) = \tilde{K}\left(1+\frac{d_{12}}{a_1}\right)^{M_1}\left(1-\frac{d_{12}}{a_2}\right)^{M_2} -a_1 < d_{12} < a_2$$

where

$$\frac{M_1}{a_1} = \frac{M_2}{a_2}$$

$$M_1 = \tfrac{1}{2}\left\{R-2-R(R+2)\sqrt{\frac{\beta_1}{\beta_1(R+2)^2+16(R+1)}}\right\}$$

$$M_2 = \tfrac{1}{2}\left\{R-2+R(R+2)\sqrt{\frac{\beta_1}{\beta_1(R+2)^2+16(R+1)}}\right\}$$

$$R = \frac{6(\beta_2-\beta_1-1)}{(6+3\beta_1-2\beta_2)}$$

$$a_1+a_2 = \tfrac{1}{2}\sqrt{m_2}\sqrt{\left\{\beta_1(R+2)^2+16(R+1)\right\}}$$

$$\tilde{K}^{-1} = \int_{-\infty}^{\infty} f(d_{12})dd_{12}$$

$$\kappa = \frac{\beta_1(\beta_2+3)^2}{4(4\beta_2-3\beta_1)(2\beta_2-3\beta_1-6)};$$

with

$$\beta_1 = \frac{m_3^2}{m_2^3}\text{and}\beta_2 = \frac{m_4}{m_2^2}$$

where m_2, m_3 and m_4 denote the (unconditional) variance, third and fourth central moments of d_{12}.

4.3.3 Cornish-Fisher Expansions

The Cornish–Fisher percentage points can be calculated in the following way: The standardised version of the difference in the reciprocal of the coefficient of variation d_{12} is $T = \frac{d_{12} - m'_1}{\sqrt{m_2}}$ and the percentage point of level α of T is defined as t_α. With this definition it follows that a Cornish–Fisher expansion (see Cornish and Fisher (1937) and Fisher and Cornish (1960)) for the percentage point t_α of T is given by:

$$\tilde{t}_\alpha = z_\alpha + \frac{1}{6}\ell_3(z_\alpha^2 - 1) + \frac{1}{24}\ell_4(z_\alpha^3 - 3z_\alpha) - \frac{1}{36}\ell_3^2(2z_\alpha^3 - 5z_\alpha)$$

where z_α is the corresponding percentage point of the standard normal distribution,

$\ell_r = \frac{\kappa_r}{(\kappa_2)^{r/2}} (r = 3,4)$ and κ_r is the r–th cumulant of d_{12}. Also

$$\kappa_2 = m_2, \quad \kappa_3 = m_3 \quad \text{and} \quad \kappa_4 = m_4 - 3m_2^2.$$

The percentage point of level α of d_{12} is then given by $\tilde{t}_\alpha \sqrt{m_2} + m'_1$.

5. Example and descriptive statistics

Consider the monthly prices of three commodities, gold, platinum and oil for the period January 1980 to April 2007. Let P_t be the price of a commodity at time t and let $r_t = \log\left(P_t / P_{t-1} \right)$ be the logarithmic return at time t for each commodity. This method of calculating the return inevitably assumes that prices are log normally distributed if r_t is then assumed to be normally distributed. We wish to identify the investment that has the highest performance based on the observed estimates of the reciprocal of the coefficient of variation of the logarithmic returns from a Bayesian point of view.

Table 5.1: Summary of the observed commodity data.

Commodity	Gold	Platinum	Oil
Sample size (n_i)	327	327	327
Estimated monthly mean logarithm return $(\bar{r}_i)$	0.000020	0.001357	0.001842
Estimated monthly standard deviation of the logarithm return (s_i)	0.041477	0.055984	0.080103
Estimated monthly classical reciprocal of the coefficient of variation $\widehat{\theta}_i = \frac{\bar{r}_i}{s_i} \quad i = 1,2,3$.	0.000482	0.024239	0.022995

Data sources: Gold: London gold price US$: SARB, Oil: Crude oil (petroleum); simple average of three spot prices-Dated Brent, West Texas Intermediate and Dubai Fateh US$ per barrel:IMF, Platinum: average of the London PM fix price US$:Kitco Bullion Dealers.

The following graphs show the monthly price of the commodities from January 1980 to April 2007.

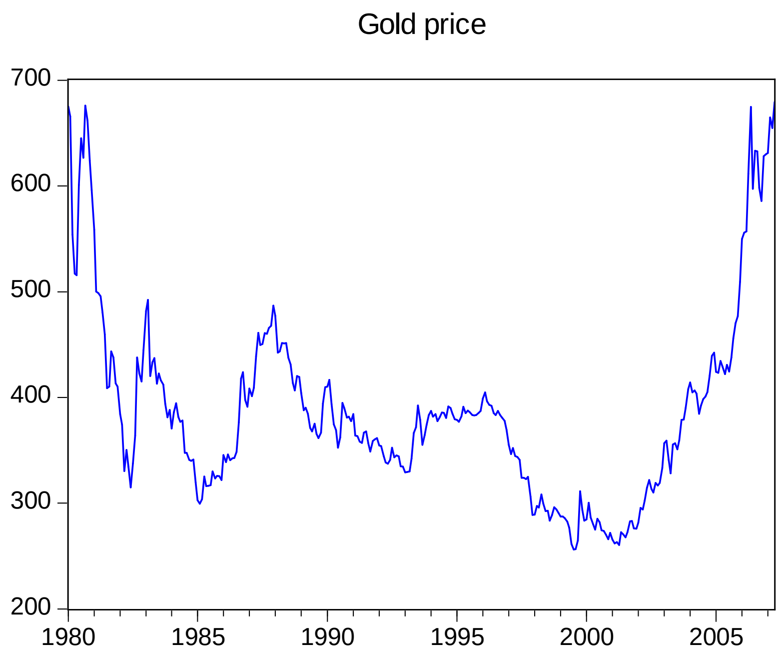

Figure 5.1: Monthly gold price from January 1980 to April 2007.

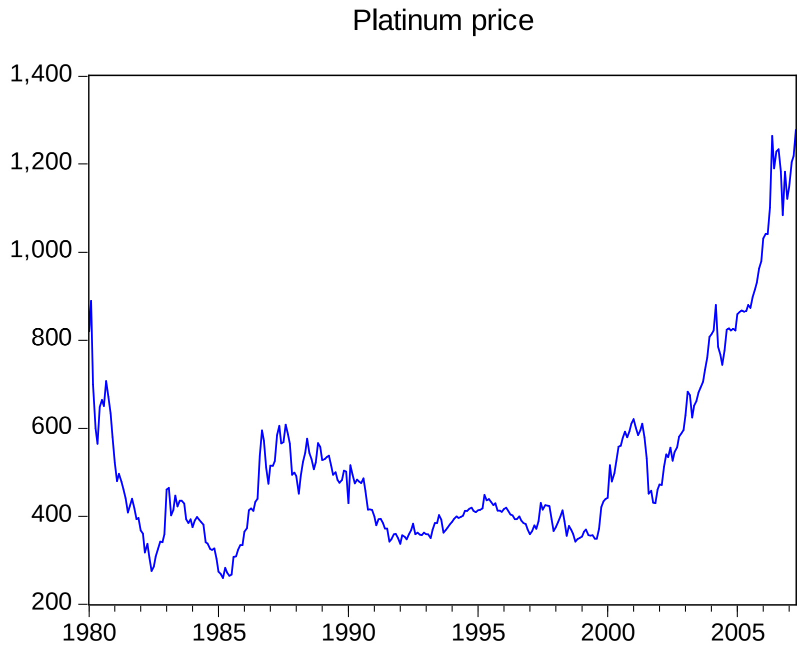

Figure 5.2: Monthly platinum price from January 1980 to April 2007.

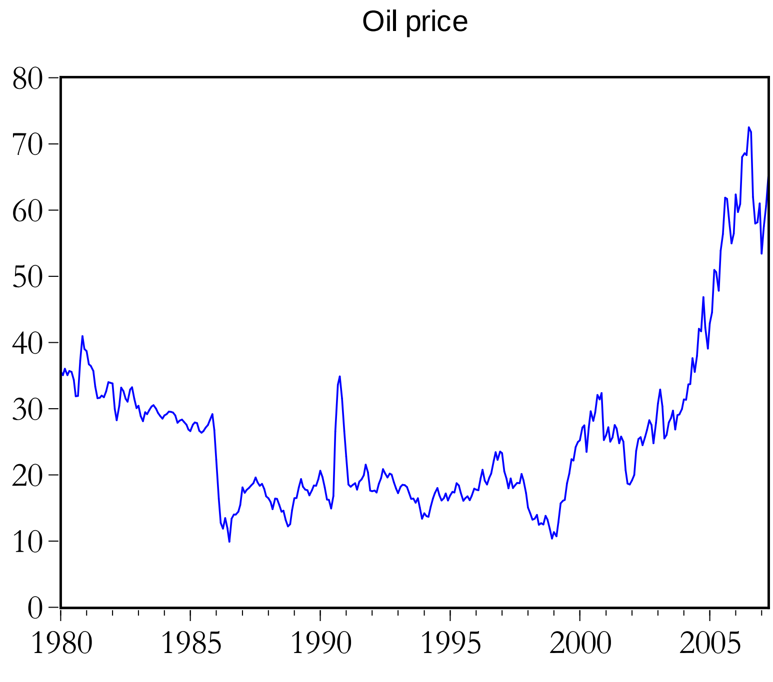

Figure 5.3: Monthly oil price from January 1980 to April 2007.

The classical estimates of the reciprocals of the coefficients of variation are generally very low for all the three commodities and suggest investing in platinum rather than gold or oil. The logarithmic return of oil has a standard deviation which is about twice that of gold suggesting that investing in oil is very risky when compared to investing in gold or platinum. The average monthly return of gold is very close to zero over this period giving rise to a very low value of the reciprocal of the coefficient of variation.

Figure 5.4 shows the logarithmic returns of the commodities on the same graph. The zero line is also indicated.

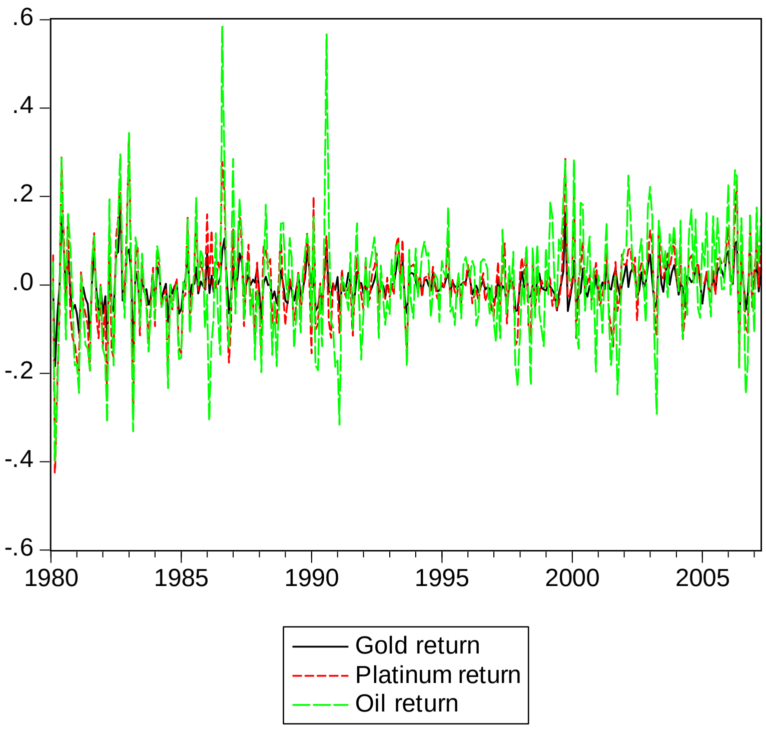

Figure 5.4: Monthly gold, platinum and oil logarithmic returns from February 1980 to April 2007.

The greater volatility in the oil return is apparent. Gold and platinum seem to have similar risk as measured by the standard deviation.

6. Results

The procedure in section 3.1 is used to simulate the reciprocal the coefficient of variation $\theta_i = \frac{\mu_i}{\sigma_i} \quad i = 1,2,3$ corresponding to gold, platinum and oil respectively. Parts of the simulated values are presented in table 6.1 below.

Table 6.1: Part of the 1000 Bayesian simulated reciprocal of the coefficients of variation values from the three commodities.

ℓ	Gold θ_1^ℓ	Platinum θ_2^ℓ	Oil θ_3^ℓ
1	0.0142	0.0968	-0.0348
2	-0.0738	0.0632	0.0014
3	0.0684	0.0112	-0.0249
4	0.0867	0.0123	-0.0603
5	0.0582	-0.0364	0.0259
6	-0.0554	-0.1428	0.0487
7	0.0341	-0.0248	0.0841
8	- 0.0064	0.0960	0.0439
⋮	⋮		⋮
⋮	⋮		⋮
1000	-0.0802	-0.0564	0.0586

The Bayesian based significance testing of equality of the reciprocal of the coefficients of variation and pair wise comparison tests among the commodities as outlined in section 3.2 and 3.3 is now used to test for equality. In this application 10 000 simulations were done. The estimated p-value is 0.9999 indicating no significant difference in the log returns of the three commodities.

Pair wise comparisons are nonetheless carried out for the three commodities.

For the simulated $\theta_i \quad i=1,2,3$ (as illustrated in the columns of table 6.1) define $d_{12}=\theta_{gold}^{(\ell)}-\theta_{oil}^{(\ell)}=\left(\frac{\mu_{gold}^{(\ell)}}{\sigma_{gold}^{(\ell)}}\right)-\left(\frac{\mu_{oil}^{(\ell)}}{\sigma_{oil}^{(\ell)}}\right)$ for $\ell=1:10000$.

Below is the histogram of simulated d_{12} values.

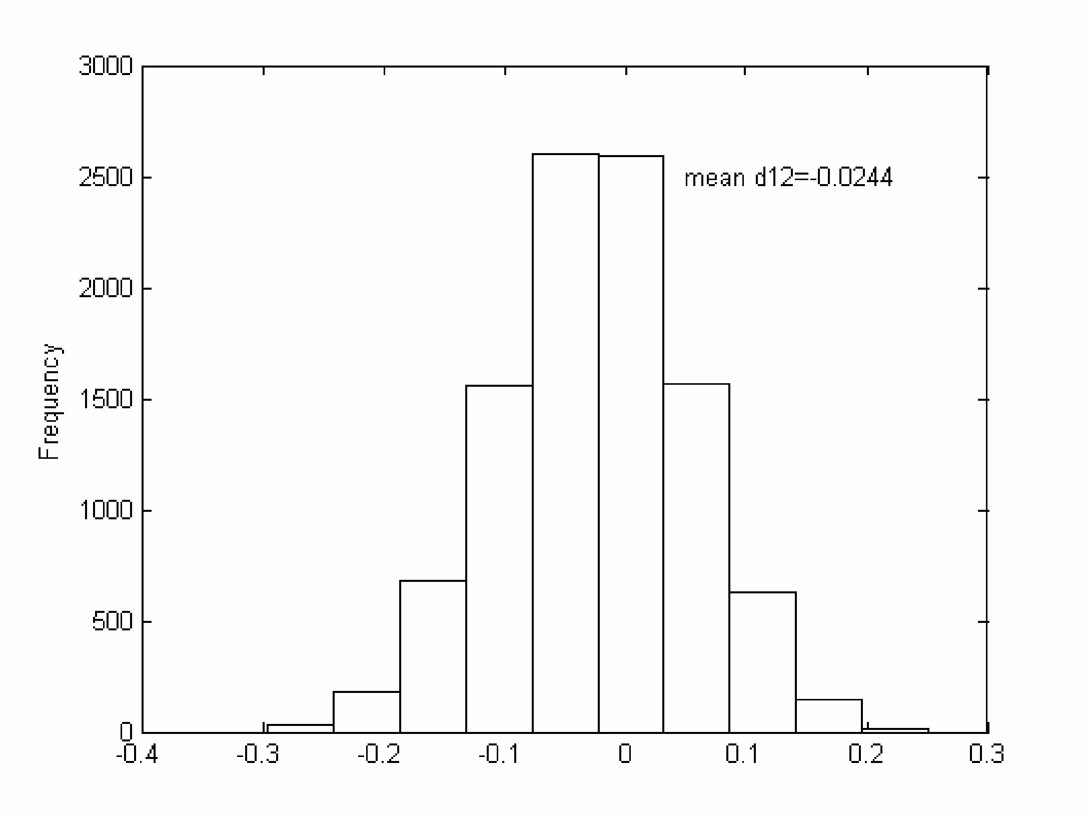

Figure 6.1: Frequency histograms of 10000 simulated differences $d_{12}{}^{*}=\theta_{gold}{}^{*}-\theta_{platinum}{}^{*}$ values from the two commodities.

Below is the histogram of simulated d_{13} values.

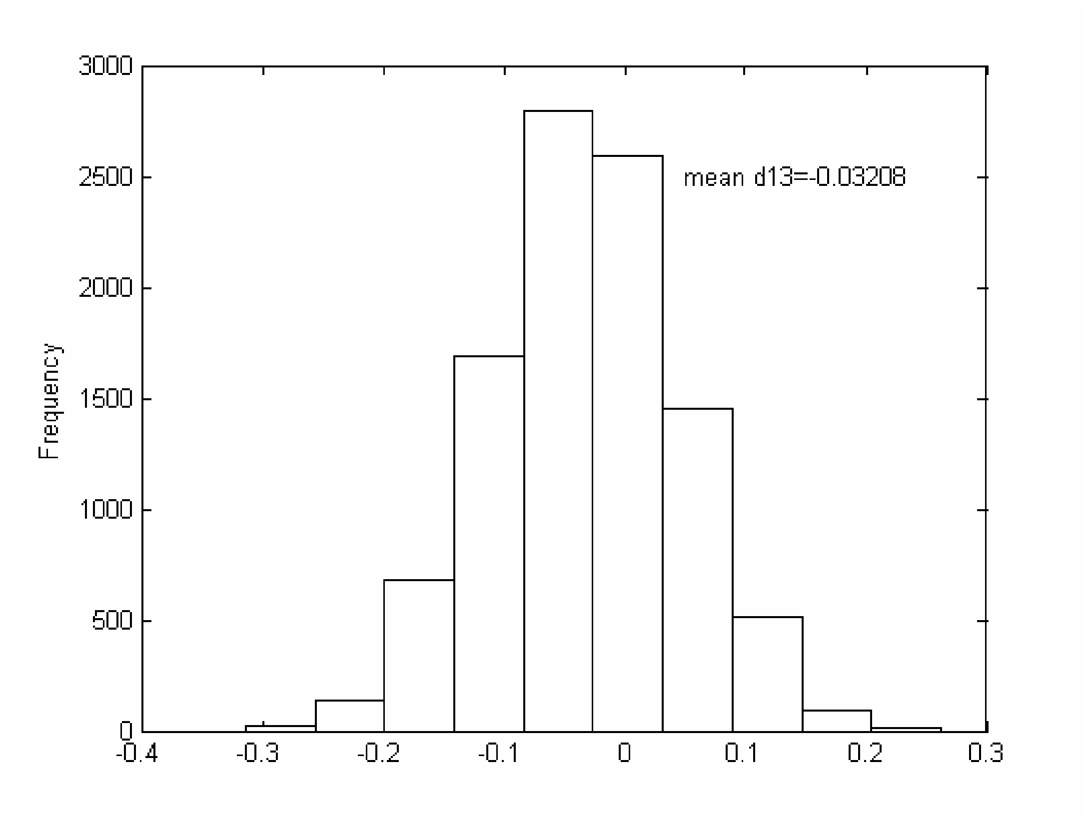

Figure 6.2: Frequency histograms of 10000 simulated differences $d_{13}^{*} = \theta_{gold}^{*} - \theta_{oil}^{*}$ *values from the two commodities.*

Below is the histogram of simulated d_{23} values.

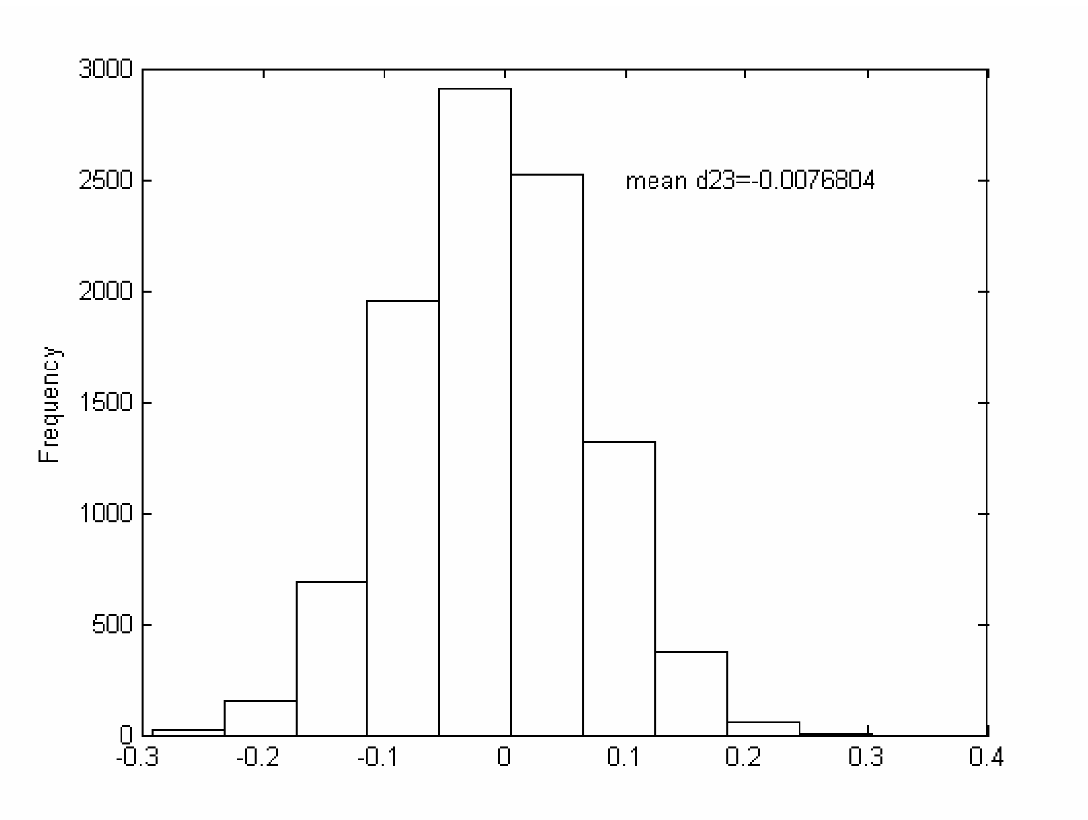

Figure 6.3: Frequency histograms of 10000 simulated differences $d_{23}^{*} = \theta_{platinum}^{*} - \theta_{oil}^{*}$ *values from the two commodities.*

To construct the percentile credibility interval for the differences $d_{12} = \theta_{gold} - \theta_{platinum}$ we sort the $d_{12}^{*(\ell)} = \theta_{gold}^{*(\ell)} - \theta_{platinum}^{*(\ell)}$ values in ascending order so that:

$$d_{12}^{*(1)} \le d_{12}^{*(2)} \le \cdots \le d_{12}^{*(1000)}$$

In this application 1000 values of d_{12} are sorted from least to greatest and the critical values are found by selecting the value in the position $(\frac{\alpha}{2}) \times 10000$ as the lower bound and the value in the position $(1-\frac{\alpha}{2})\times 10000$ as the upper bound. The credibility interval is then constructed as $d_{12}\left((\frac{\alpha}{2})\times 10000\right)-d_{12}\left((1-\frac{\alpha}{2})\times 10000\right)$. The 95% credibility interval is $d_{12}(250)-d_{12}(9750)$. The decision rule using these intervals to test $H_o:\frac{\mu_{gold}}{\sigma_{gold}}=\frac{\mu_{platinum}}{\sigma_{platinum}}$ vs. $H_1:\frac{\mu_{gold}}{\sigma_{gold}}\neq\frac{\mu_{platinum}}{\sigma_{platinum}}$ is as follows: Reject H_o if zero is not contained in the interval mentioned above, otherwise, accept H_o.

A similar procedure is followed for d_{13} and d_{23}.

Table 6.2: Results of the pair-wise comparisons of θ.

Test Pair	Observed d_{ik}	Simulated Bayesian mean d_{ik}	95% credibility interval
Gold vs. Platinum	-0.023757	-0.02440	(-0.1803;0.1291)
Gold vs. Oil	-0.022513	-0.032080	(-0.1834;0.1227)
Platinum vs. Oil	0.001244	-0.007680	(-0.16409;0.1427)

The reciprocal of the coefficients of variation of gold and oil do not significantly differ from one another as zero is included in the interval. The higher returns in oil are offset by the greater volatility. The other comparisons among the commodities give similar results.

A Pearson type 1 curve for the difference $d_{12}=\theta_{gold}-\theta_{platinum}$ is given in figure 6. 4. As mentioned, the advantage of the Pearson curve is that it's a formula which can then be used to plot the theoretical distribution.

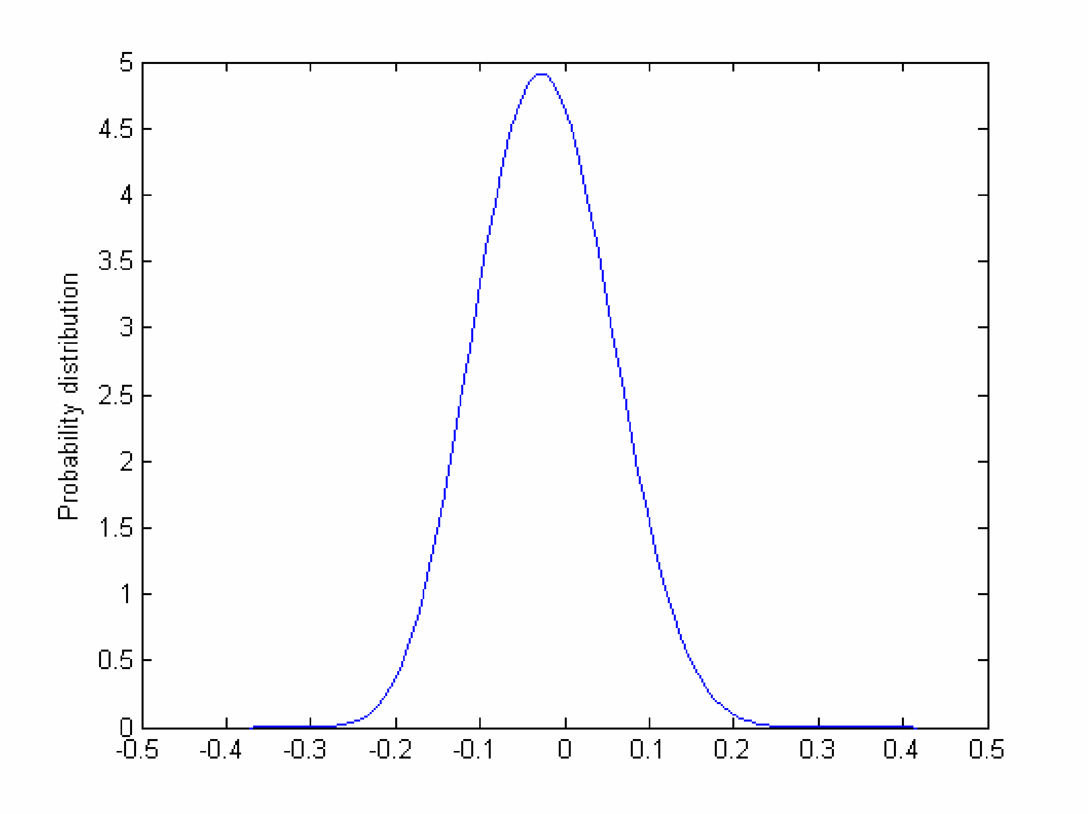

Figure 6.4: Pearson type 1 curve for the difference $d_{12} = \theta_{gold} - \theta_{platinum}$ *of the two commodities.*

A Pearson type 1 curve for the difference $d_{13} = \theta_{gold} - \theta_{oil}$ is given in figure 6.5.

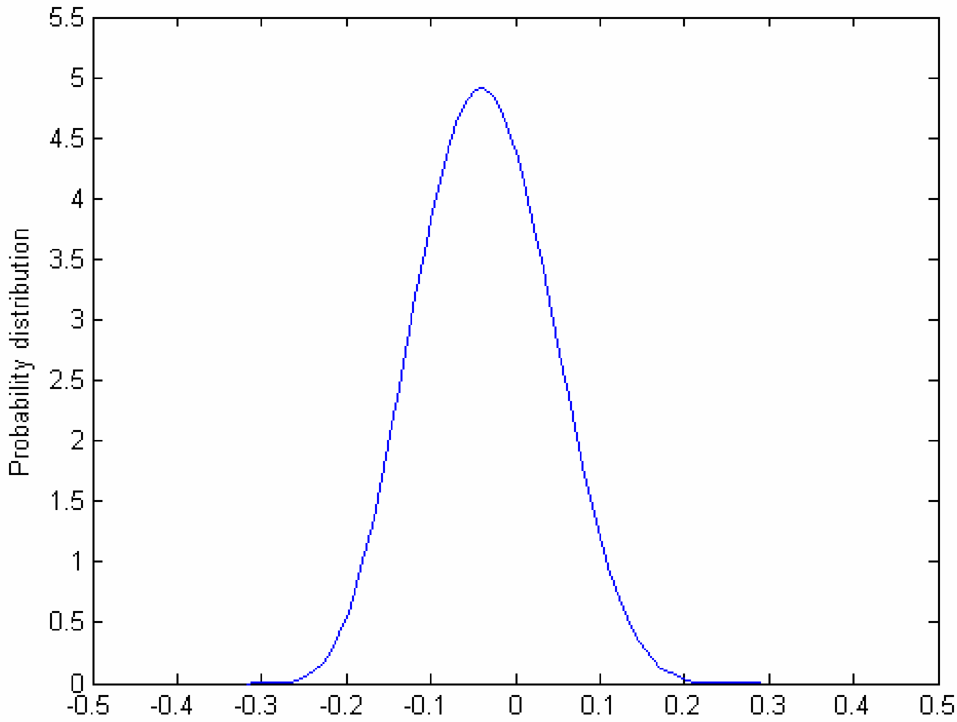

Figure 6.5: Pearson type 1 curve for the difference $d_{13} = \theta_{gold} - \theta_{oil}$ *of the two commodities.*

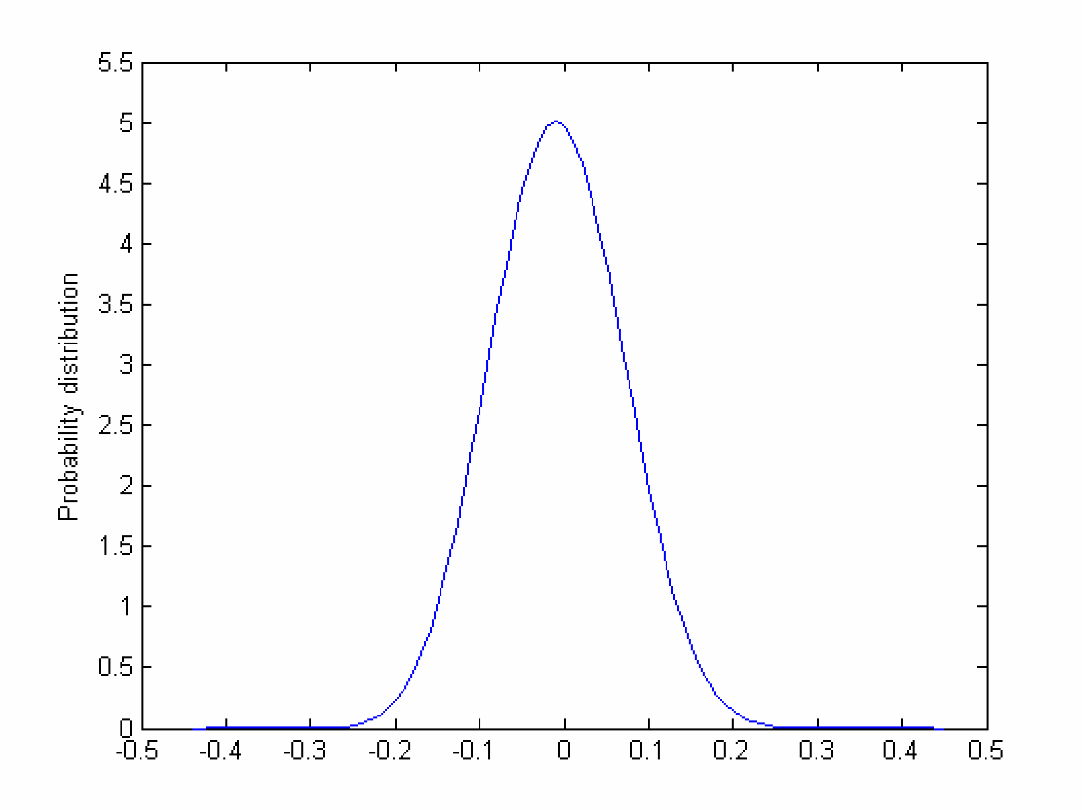

Figure 6.6: Pearson type 1 curve for the difference $d_{23} = \theta_{platinum} - \theta_{oil}$ *of the two commodities.*

Zero is included in the 95% credibility interval in all three cases indicating that there are no significant differences in the reciprocals of the coefficients of variation among the three commodities over the stated period.

The problem of selecting the best commodity can also be looked at from a ranking and selection perspective. In the past 30 years, beginning with the fundamental papers of Bechhofer (1954) and Gupta (1956), ranking and selection procedures have been developed to overcome the inadequacy of testing procedures. From a Bayesian point of view, ranking and selection is quite simple. To calculate the probability that oil, say, performed better than gold, we assign a rank to each simulation in a row (table 6.1). The highest of the θ_i^{ℓ} in the row is assigned a rank of 1, and the lowest value is assigned a rank of 3, the other value necessarily gets the rank of 2. The ranks are shown in table 6.3 below.

Table 6.3: Part of the 1000 rankings of the Bayesian simulated coefficients of variation values from the three commodities.

ℓ	Gold Ranking	Platinum Ranking	Oil Ranking
1	2	1	3
2	3	1	2
3	1	2	3
4	1	2	3
5	1	3	2
6	2	3	1
7	2	3	1
8	3	1	2
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮
1000	3	2	1

The ranking results are now summarized in table 6.4 below.

Table 6.4: Summary of the 1000 rankings of the Bayesian simulated coefficients of variation values.

Frequency of	Gold θ_1	Platinum θ_2	Oil θ_3
1's	205	353	442
2's	299	369	332
3's	496	278	226

The required probabilities can now be calculated by dividing the above frequencies by 1000. The probabilities are given in table 6.5 below.

Table 6.5: Probabilities that a given commodity is ranked 1st, 2nd or 3rd in terms of return per unit of risk.

Probabilities	Gold θ_1	Platinum θ_2	Oil θ_3
Prob $(rank\ \theta_i = 1)$	0.205	0.353	0.442
Prob $(rank\ \theta_i = 2)$	0.299	0.369	0.332
Prob $(rank\ \theta_i = 3)$	0.496	0.278	0.226

Whilst oil has the highest probability (0.442) of being ranked 1st using the reciprocal of the coefficient of variation of the logarithmic returns, there is almost an equal chance that that it will be outperformed by a portfolio in gold and platinum. Platinum has the highest probability (0.369) of being ranked 2nd and gold has the highest probability (0.496) of being ranked 3rd.

7. Conclusion

In this paper, the reciprocal of the coefficient of variation $\theta = \frac{\mu}{\sigma}$ is used to compare the returns of three commodities, namely gold, platinum and oil, using a Bayesian approach. Results indicate that oil has a higher probability to outperform the other two commodities but there are no significant differences in the reciprocals of the coefficients of variation among the three commodities according to the credibility intervals (Bayesian confidence intervals) over the period from January 1980 to April 2007.

Bayesian inference has a number of advantages. A full Bayesian analysis provides a natural way of taking into account all sources of uncertainty in the estimation of the parameters. In the paper, uncertainty about the true value of the reciprocal of the coefficient of variation is incorporated into the analysis through the choice of a non informative prior distribution.

Appendix

Proof of equation 2.3

The likelihood of n_i independent and identically normally distributed random variables is:

$$l(\mu_i,\sigma_i^2\,|\,\underline{r_{i.}}) \propto p(\underline{r_{i.}}\,|\,\mu_i,\sigma_i^2)$$

$$= \prod_{j=1}^{n_i}\left(2\pi\sigma_i^{\ 2}\right)^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}\left(r_{ij}-\mu_i\right)^2\Big/\sigma_i^{\ 2}\right\}$$

$$=\left(\sigma_i^{\ 2}\right)^{-n_i/2}\exp\left(-\frac{1}{2}\sum\left(r_{ij}-\overline{r_i}+\overline{r_i}-\mu_i\right)^2\Big/\sigma_i^{\ 2}\right)$$

$$=\left(\sigma_i^{\ 2}\right)^{-n_i/2}\exp\left(-\frac{1}{2}\left\{\sum\left(r_{ij}-\overline{r_i}\right)^2+n_i\left(\overline{r_i}-\mu_i\right)^2\right\}\Big/\sigma_i^{\ 2}\right)$$

$$=\left(\sigma_i^{\ 2}\right)^{-n_i/2}\exp\left(-\frac{1}{2}\left\{S_i+n_i\left(\overline{r_i}-\mu_i\right)^2\right\}\Big/\sigma_i^{\ 2}\right)$$

where $S_i=\sum\left(r_{ij}-\overline{r_i}\right)^2$.

It is convenient to define $s_i^2=\dfrac{S_i}{n_i-1}$.

If we take the vague prior, then

$$p(\mu_i,\sigma_i^2\,|\,\underline{r_{i.}}) \propto p(\mu_i,\sigma_i^{\ 2})\,p(\underline{r_{i.}}\,|\,\mu_i,\sigma_i^2)$$

$$\propto \left(\sigma_i^{\ 2}\right)^{-1}\left(\sigma^2\right)^{-n_i/2}\exp\left(-\frac{1}{2}\left\{S_i+n_i\left(\overline{r_i}-\mu_i\right)^2\right\}\Big/\sigma_i^{\ 2}\right)$$

$$\propto \left(\sigma_i^{\ 2}\right)^{-n_i/2-1}\exp\left(-\frac{1}{2}\left\{S_i+n_i\left(\overline{r_i}-\mu_i\right)^2\right\}\Big/\sigma_i^{\ 2}\right)$$

For reasons which will appear later it is convenient to set

$$v_i=n_i-1$$

in the power of σ_i^2, but not in the exponential, so that

$$p(\mu_i, \sigma_i^2 \left| \underline{r_{i.}} \right.) \propto \left(\sigma_i^2\right)^{-(v_i+1)/2-1} \exp\left(-\frac{1}{2}\left\{S_i + n_i\left(\overline{r_i} - \mu_i\right)^2\right\} \Big/ \sigma_i^2\right)$$

Marginal distribution of the variance

If knowledge about σ_i^2 rather than μ_i is required, μ_i is integrated out from the posterior distribution.

$$p(\sigma_i^2 \left| \underline{r_{i.}} \right.) = \int p(\mu_i, \sigma_i^2 \mid \underline{r_{i.}}) d\mu_i$$

$$\propto \int_{-\infty}^{\infty} \left(\sigma_i^2\right)^{-\frac{n_i}{2}-1} \exp\left(-\frac{1}{2}\left\{S_i + n_i\left(\mu_i - \overline{r_i}\right)^2\right\} \Big/ \sigma_i^2\right) d\mu_i$$

$$\propto \int_{-\infty}^{\infty} \left(\sigma_i^2\right)^{-\frac{n_i}{2}-1} \exp\left(-\frac{1}{2}\left\{S_i + n_i\left(\mu_i - \overline{r_i}\right)^2\right\} \Big/ \sigma_i^2\right) d\mu_i$$

$$\propto \left(\sigma_i^2\right)^{-(\frac{n_i}{2}+\frac{1}{2})} \exp\left(-\frac{1}{2} S_i / \sigma_i^2\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{-\frac{1}{2}\left(\mu_i - \overline{r_i}\right)^2 \Big/ \left(\sigma_i^2/n_i\right)\right\} d\mu_i$$

$$\propto \left(\sigma_i^2\right)^{-(\frac{n_i-1}{2}+1)} \exp\left(-\tfrac{S_i}{2} / \sigma_i^2\right)$$

$$= \left(\sigma_i^2\right)^{-(v_i/2+1)} \exp\left(-\tfrac{S_i}{2} / \sigma_i^2\right), \text{ where } v_i = n_i - 1$$

as the last integral is that of a normal density.

It follows that the posterior density of the variance is $IG\left(\frac{v_i}{2}, \frac{S_i}{2}\right)$. Except for the fact that n_i is replaced by $v_i = n_i - 1$

Conditional density of the mean for given variance

The joint posterior can be written in the following form

$$p(\mu_i, \sigma_i^2 \left| \underline{r_{i.}} \right.) = p(\sigma_i^2 \mid \underline{r_{i.}})\, p(\mu_i \mid \sigma_i^2, \underline{r_{i.}})$$

Thus

$$p(\mu_i \mid \sigma_i^2, \underline{r_{i.}}) = \frac{p(\mu_i, \sigma_i^2 \mid \underline{r_{i.}})}{p(\sigma_i^2 \mid \underline{r_{i.}})}$$

$$p(\mu_i,\sigma_i^2 \,|\, \underline{r_{i.}}) \propto \left(\sigma_i^2\right)^{-(\nu_i+1)/2-1} \exp\left(-\frac{1}{2}\left\{S_i + n_i\left(\overline{r_i}-\mu_i\right)^2\right\}\Big/\sigma_i^2\right)$$

$$p(\sigma_i^2 \,|\, \underline{r_{i.}}) \propto \left(\sigma_i^2\right)^{-\nu_i/2-1} \exp\left(-\frac{1}{2}S_i\big/\sigma_i^2\right)$$

This implies that

$$p(\mu_i \,|\, \sigma_i^2, \underline{r_{i.}}) \propto \left(\sigma_i^2\right)^{-1/2} \exp\left(-\frac{1}{2}n_i(\mu_i-\overline{r_i})^2\big/\sigma_i^2\right)$$

which as the density integrates to unity implies that

$$p(\mu_i \,|\, \sigma_i^2, \underline{r_{i.}}) = \frac{1}{\sqrt{2\pi\sigma_i^2/n_i}} \exp\left(-\frac{1}{2}(\mu_i-\overline{r_i})^2\Big/\left(\sigma_i^2/n_i\right)\right)$$

that is, for given σ_i^2 and $\underline{r_{i.}}$, the distribution of the mean μ_i is $N\left(\overline{r_i},\sigma_i^2/n_i\right)$.

Proof of theorem 3.1

Since $\mu \,|\, \sigma^2, \underline{r} \sim N\left(\overline{r},\dfrac{\sigma^2}{n}\right)$ and

$\kappa = \dfrac{\nu s^2}{\sigma^2} \sim \chi^2_\nu$, it follows that

$\theta \,\big|\, \hat{\theta}, \kappa \sim N\left(a\sqrt{\kappa}, \dfrac{1}{n}\right)$

where $a = \dfrac{\hat{\theta}}{\sqrt{\nu}}$

Therefore

$$p\left(\theta \,\middle|\, \hat{\theta}\right) = \int_0^\infty p\left(\theta \,\middle|\, \hat{\theta}, \kappa\right) p(\kappa)\, d\kappa$$

$$= \frac{\sqrt{n}}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{n}{2}(\theta-a\sqrt{\kappa})^2\right) \kappa^{-\frac{1}{2}\nu-1} \exp\left(-\frac{\kappa}{2}\right) d\kappa$$

Now

$$\exp\left(-\frac{n}{2}(\theta - a\sqrt{\kappa}\,)^2\right) = \exp\left(-\frac{n}{2}(\theta^2 - 2\theta a\sqrt{\kappa} + a^2\kappa)\right)$$

$$= \exp\left(-\frac{n\theta^2}{2} + na\theta\sqrt{\kappa} - \frac{na^2\kappa}{2}\right)$$

Since

$$\exp\left(na\theta\sqrt{\kappa}\,\right) = \sum_{l=0}^{\infty}\frac{\left(n\theta a\sqrt{\kappa}\right)^l}{l!}$$

it follows

$$p\left(\theta\middle|\hat{\theta}\right) = \frac{\sqrt{n}\exp\left(-\frac{1}{2}n\theta^2\right)}{2^{\frac{\nu}{2}}\sqrt{2\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\int_0^{\infty}\kappa^{\frac{1}{2}\nu-1}\left(\sum_{l=0}^{\infty}\frac{nta\sqrt{\kappa}}{l!}\right)^l \exp\left(-\frac{\kappa}{2}\left(1+na^2\right)\right)d\kappa$$

$$\int_0^{\infty}\kappa^{\frac{1}{2}(\nu+l)-1}\exp\left(-\frac{1}{2}\kappa\left(1+na^2\right)\right)d\kappa = \frac{2^{\frac{1}{2}(\nu+l)}\,\Gamma\left(\frac{\nu+l}{2}\right)}{\left(1+na^2\right)^{\frac{1}{2}(\nu+l)}}$$

and substituting $a = \dfrac{\hat{\theta}}{\sqrt{\nu}}$, the posterior distribution of $\theta = \dfrac{\mu}{\sigma}$ is

$$p\left(\theta\middle|\hat{\theta}\right) = \frac{\sqrt{n}\exp\left(-\frac{n\theta^2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{2\pi}}\sum_{l=0}^{\infty}\left(\frac{n\theta\hat{\theta}}{\sqrt{\nu}}\right)^l \frac{\Gamma\left(\frac{\nu+l}{2}\right)2^{\frac{1}{2}l}}{l!\left(1+\frac{n}{\nu}\hat{\theta}^2\right)^{\frac{1}{2}(\nu+l)}} \quad -\infty < \theta < \infty\,.$$

Proof of theorem 4.1

From section 4.3.1

$$\mu'_1 = \bar{r}_1\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 {s_1}^2}} - \bar{r}_2\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 {s_2}^2}}$$

$$\mu_2 = \left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

and now

$$\mu'_2 = \mu_2 + \left(\mu'_1\right)^2 = \left(\frac{1}{n_1} + \frac{1}{n_2}\right) + \left(\frac{(\bar{r}_1)^2(\chi^2_{\nu_1})}{(\nu_1 {s_1}^2)} - 2\bar{r}_1\bar{r}_2\sqrt{\frac{\chi^2_{\nu_1}\chi^2_{\nu_2}}{\nu_1 {s_1}^2 \nu_2 {s_2}^2}} + \frac{(\bar{r}_2)^2(\chi^2_{\nu_2})}{(\nu_2 {s_2}^2)}\right)$$

$$\mu'_3 = \mu_3 + 3\mu_2\mu'_1 + (\mu'_1)^3$$

$$= 0 + 3\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left(\bar{r}_1\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 s_1{}^2}} - \bar{r}_2\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2{}^2}}\right) + \left(\bar{r}_1\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 s_1{}^2}} - \bar{r}_2\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2{}^2}}\right)^3$$

$$= 3\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left(\bar{r}_1\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 s_1{}^2}} - \bar{r}_2\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2{}^2}}\right) + \frac{\bar{r}_1^{\,3}(\chi^2_{\nu_1})^{\frac{3}{2}}}{(\nu_1 s_1{}^2)^{\frac{3}{2}}}$$

$$-3\frac{(\bar{r}_1)^2(\chi^2_{\nu_1})}{(\nu_1 s_1{}^2)}\frac{(\bar{r}_2)(\chi^2_{\nu_2})^{\frac{1}{2}}}{(\nu_2 s_2{}^2)^{\frac{1}{2}}} + 3\frac{(\bar{r}_1)(\chi^2_{\nu_1})^{\frac{1}{2}}}{(\nu_1 s_1{}^2)^{\frac{1}{2}}}\frac{(\bar{r}_2)^2(\chi^2_{\nu_2})}{(\nu_2 s_2{}^2)} - \frac{(\bar{r}_2)^3(\chi^2_{\nu_2})^{\frac{3}{2}}}{(\nu_2 s_2{}^2)^{\frac{3}{2}}}$$

$$\mu'_4 = \mu_4 + 4\mu'_1\mu_3 + 6(\mu'_1)^2\mu_2 + (\mu'_1)^4$$

$$= 3\left\{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\right\}^2 + 0 + 6\left(\bar{r}_1\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 s_1{}^2}} - \bar{r}_2\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2{}^2}}\right)^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)$$

$$+\left(\bar{r}_1\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 s_1{}^2}} - \bar{r}_2\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2{}^2}}\right)^4$$

$$= 3\left\{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\right\}^2 + 0 + 6\left(\frac{(\bar{r}_1)^2(\chi^2_{\nu_1})}{(\nu_1 s_1{}^2)} - 2\bar{r}_1\bar{r}_2\sqrt{\frac{\chi^2_{\nu_1}\chi^2_{\nu_2}}{\nu_1 s_1{}^2\nu_2 s_2{}^2}} + \frac{(\bar{r}_2)^2(\chi^2_{\nu_2})}{(\nu_2 s_2{}^2)}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)$$

$$+\frac{(\bar{r}_1)^4(\chi^2_{\nu_1})^2}{(\nu_1 s_1{}^2)^2} - 4\frac{(\bar{r}_1)^3(\bar{r}_2)(\chi^2_{\nu_1})^{\frac{3}{2}}}{(\nu_1 s_1{}^2)^{\frac{3}{2}}}\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2{}^2}} + 6\frac{(\bar{r}_1)^2(\chi^2_{\nu_1})}{(\nu_1 s_1{}^2)}\frac{(\bar{r}_2)^2(\chi^2_{\nu_2})}{(\nu_2 s_2{}^2)}$$

$$-4(\bar{r}_1)\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 s_1{}^2}}\frac{(\bar{r}_2)^3(\chi^2_{\nu_2})^{\frac{3}{2}}}{(\nu_2 s_2{}^2)^{\frac{3}{2}}} + \frac{(\bar{r}_2)^4(\chi^2_{\nu_2})^2}{(\nu_2 s_2{}^2)^2}$$

$$= 3\left(\frac{1}{n_1}+\frac{1}{n_2}\right)^2 + 6\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left(\frac{(\bar{r}_1)^2(\chi^2_{\nu_1})}{(\nu_1 s_1{}^2)} - 2\bar{r}_1\bar{r}_2\sqrt{\frac{\chi^2_{\nu_1}\chi^2_{\nu_2}}{\nu_1 s_1{}^2\nu_2 s_2{}^2}} + \frac{(\bar{r}_2)^2(\chi^2_{\nu_2})}{(\nu_2 s_2{}^2)}\right)$$

$$+\frac{(\bar{r}_1)^4(\chi^2_{\nu_1})^2}{(\nu_1 s_1{}^2)^2} - 4\frac{(\bar{r}_1)^3(\bar{r}_2)(\chi^2_{\nu_1})^{\frac{3}{2}}}{(\nu_1 s_1{}^2)^{\frac{3}{2}}}\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2{}^2}} + 6\frac{(\bar{r}_1)^2(\bar{r}_2)^2(\chi^2_{\nu_1})(\chi^2_{\nu_2})}{(\nu_1 s_1{}^2)(\nu_2 s_2{}^2)}$$

$$-4(\overline{r}_1)(\overline{r}_2)^3 \frac{(\chi^2_{\nu_1})^{\frac{1}{2}}(\chi^2_{\nu_2})^{\frac{3}{2}}}{(\nu_1 s_1{}^2)^{\frac{1}{2}}(\nu_2 s_2{}^2)^{\frac{3}{2}}} + \frac{(\overline{r}_2)^4(\chi^2_{\nu_2})^2}{(\nu_2 s_2{}^2)^2}$$

Proof of theorem 4.2

Since the inverse gamma distribution is given by

$$IG(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp(-\beta/x)$$

with

$$\int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp(-\beta/x) dx = 1$$

and therefore

$$\int_0^\infty x^{-\alpha-1} \exp(-\beta/x) = \frac{\Gamma(\alpha)}{\beta^\alpha}$$

For the inverse gamma in 2.3 and 2.4 $\beta_i = \frac{1}{2}(n_i - 1)s_i{}^2 = \frac{\nu_i s_i{}^2}{2}$ and $\alpha_i = \frac{1}{2}(n_i - 1) = \frac{\nu_i}{2}$.

this integral is $\dfrac{\Gamma\left(\frac{\nu_i}{2}\right)}{\left\{\frac{\nu_i s_i{}^2}{2}\right\}^{\frac{\nu_i}{2}}} = \dfrac{\Gamma\left(\frac{\nu_i}{2}\right)}{\left\{\frac{\nu_i s_i{}^2}{2}\right\}^{\frac{\nu_{i-1}}{2}}\left\{\frac{\nu_i s_i{}^2}{2}\right\}^{\frac{1}{2}}} = \dfrac{\Gamma\left(\frac{\nu_i}{2}\right)}{\Gamma\left(\frac{\nu_1+1}{2}\right)\left\{\frac{\nu_i s_i{}^2}{2}\right\}^{\frac{1}{2}}}$

$$\begin{aligned}
E(d_{12} \mid \underline{r_i}) &= E_{\chi^2_{\nu_1}}\left(\overline{r}_1\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 s_1{}^2}}\right) - E_{\chi^2_{\nu_2}}\left(\overline{r}_2\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2{}^2}}\right) \\
&= \left(\overline{r}_1 \frac{\sqrt{2}\Gamma\left(\frac{\nu_1+1}{2}\right)}{\sqrt{\nu_1 s_1{}^2}\Gamma\left(\frac{\nu_1}{2}\right)} - \overline{r}_2 \frac{\sqrt{2}\Gamma(\left(\frac{\nu_2+1}{2}\right)}{\sqrt{\nu_2 s_2{}^2}\Gamma\left(\frac{\nu_2}{2}\right)}\right) \\
&= \sqrt{2}\left(\frac{\overline{r}_1}{\sqrt{\nu_1 s_1{}^2}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)} - \frac{\overline{r}_2}{\sqrt{\nu_2 s_2{}^2}}\frac{\Gamma(\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)
\end{aligned}$$

$$Var(d_{12} \mid \underline{r_i}) = E_{\chi^2_{\nu_1}\chi^2_{\nu_2}}(Var(d_{12} \mid \underline{r_i})) + Var_{\chi^2_{\nu_1}\chi^2_{\nu_2}}(E(d_{12} \mid \underline{r_i}))$$

$$=\left(\frac{1}{n_1}+\frac{1}{n_2}\right)+Var_{\chi^2_{\nu_1}}\left(\frac{\overline{r_1}}{\nu_1 {s_1}^2}(\sqrt{\chi^2_{\nu_1}})+\frac{\overline{r_2}}{\nu_2 {s_2}^2}(\sqrt{\chi^2_{\nu_2}})\right)$$

$$=\left(\frac{1}{n_1}+\frac{1}{n_2}\right)+\left(\frac{(\overline{r_1})^2}{\nu_1 {s_1}^2}Var(\sqrt{\chi^2_{\nu_1}})\right)+\left(\frac{(\overline{r_2})^2}{\nu_2 {s_2}^2}Var(\sqrt{\chi^2_{\nu_2}})\right)$$

$$=\left(\frac{1}{n_1}+\frac{1}{n_2}\right)+\left(\frac{(\overline{r_1})^2}{\nu_1 {s_1}^2}\left\{\nu_1-\frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right\}+\frac{(\overline{r_2})^2}{\nu_2 {s_2}^2}\left\{\nu_2-\frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}\right)$$

$$m'_3=m_3+3m_2m'_1+(m'_1)^3$$

$$m'_3=0+3\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left(\overline{r_1}\sqrt{\frac{E(\chi^2_{\nu_1})}{\nu_1 {s_1}^2}}-\overline{r_2}\sqrt{\frac{E(\chi^2_{\nu_2})}{\nu_2 {s_2}^2}}\right)+\frac{(\overline{r_1})^3E(\chi^2_{\nu_1})^{\frac{3}{2}}}{(\nu_1 {s_1}^2)^{\frac{3}{2}}}$$

$$-\frac{3(\overline{r_1})^2E(\chi^2_{\nu_1})}{(\nu_1 {s_1}^2)}\frac{(\overline{r_2})E(\chi^2_{\nu_2})^{\frac{1}{2}}}{(\nu_2 {s_2}^2)^{\frac{1}{2}}}+\frac{3(\overline{r_1})E(\chi^2_{\nu_1})^{\frac{1}{2}}}{(\nu_1 {s_1}^2)^{\frac{1}{2}}}\frac{(\overline{r_2})^2E(\chi^2_{\nu_2})}{(\nu_2 {s_2}^2)}-\frac{(\overline{r_2})^3E(\chi^2_{\nu_2})^{\frac{3}{2}}}{(\nu_2 {s_2}^2)^{\frac{3}{2}}}$$

$$m'_3=3\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left(\frac{\overline{r_1}}{\sqrt{\nu_1 {s_1}^2}}\frac{\sqrt{2}\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}-\frac{\overline{r_2}}{\sqrt{\nu_2 {s_2}^2}}\frac{\sqrt{2}\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)+\frac{(\overline{r_1})^3 2^{\frac{3}{2}}}{(\nu_1 {s_1}^2)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_1+3}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}$$

$$-\frac{3(\overline{r_1})^2\nu_1}{(\nu_1 {s_1}^2)}\frac{(\overline{r_2})}{(\nu_2 {s_2}^2)^{\frac{1}{2}}}\frac{\sqrt{2}\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}+\frac{3(\overline{r_1})}{(\nu_1 {s_1}^2)^{\frac{1}{2}}}\frac{\sqrt{2}\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{(\overline{r_2})^2\nu_2}{(\nu_2 {s_2}^2)}-\frac{(\overline{r_2})^3}{(\nu_2 {s_2}^2)^{\frac{3}{2}}}\frac{2^{\frac{3}{2}}\Gamma\left(\frac{\nu_2+3}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$=3\sqrt{2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left(\frac{\overline{r_1}}{\sqrt{\nu_1 {s_1}^2}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}-\frac{\overline{r_2}}{\sqrt{\nu_2 {s_2}^2}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)+\frac{2\sqrt{2}(\overline{r_1})^3}{(\nu_1 {s_1}^2)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_1+3}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}$$

$$-\frac{3\sqrt{2}(\overline{r_1})^2\nu_1}{(\nu_1 {s_1}^2)}\frac{(\overline{r_2})}{(\nu_2 {s_2}^2)^{\frac{1}{2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}+\frac{3\sqrt{2}(\overline{r_1})}{(\nu_1 {s_1}^2)^{\frac{1}{2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{(\overline{r_2})^2\nu_2}{(\nu_2 {s_2}^2)}-\frac{2\sqrt{2}(\overline{r_2})^3}{(\nu_2 {s_2}^2)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_2+3}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}$$

$m_3=m'_3-3m_2m'_1-(m'_1)^2$ of which

$$-3m_2m'_1 = -3\left\langle\left(\frac{1}{n_1}+\frac{1}{n_2}\right)+\left(\frac{(\bar{r}_1)^2}{\nu_1 s_1^{\,2}}\left\{\nu_1-\frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right\}+\frac{(\bar{r}_2)^2}{\nu_2 s_2^{\,2}}\left\{\nu_2-\frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}\right)\right\rangle$$

$$\sqrt{2}\left(\frac{\bar{r}_1}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}-\frac{\bar{r}_2}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)$$

$$= -3\sqrt{2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left(\frac{\bar{r}_1}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}-\frac{\bar{r}_2}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)$$

$$-3\sqrt{2}\left(\frac{(\bar{r}_1)^2}{\nu_1 s_1^{\,2}}\left(\frac{\bar{r}_1}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}-\frac{\bar{r}_2}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma(\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)\left\{\nu_1-\frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right\}\right)$$

$$-3\sqrt{2}\,\frac{(\bar{r}_2)^2}{\nu_2 s_2^{\,2}}\left(\frac{\bar{r}_1}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}-\frac{\bar{r}_2}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma(\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)\left\{\nu_2-\frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}$$

$$-(m'_1)^2 = -2\left(\frac{\bar{r}_1}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}-\frac{\bar{r}_2}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)^2$$

$$=\left(-2\frac{\left(\bar{r}_1\right)^2}{\nu_1 s_1^{\,2}}\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}-2\frac{\left(\bar{r}_2\right)^2}{\nu_2 s_2^{\,2}}\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}+4\frac{\bar{r}_1}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\bar{r}_2}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)$$

$$\therefore m_3 = 3\sqrt{2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left(\frac{\bar{r}_1}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}-\frac{\bar{r}_2}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)+\frac{2\sqrt{2}(\bar{r}_1)^3}{(\nu_1 s_1^{\,2})^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_1+3}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}$$

$$-\frac{3\sqrt{2}(\bar{r}_1)^2\nu_1}{(\nu_1 s_1^{\,2})}\frac{(\bar{r}_2)}{(\nu_2 s_2^{\,2})^{\frac{1}{2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}+\frac{3\sqrt{2}(\bar{r}_1)}{(\nu_1 s_1^{\,2})^{\frac{1}{2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{(\bar{r}_2)^2\nu_2}{(\nu_2 s_2^{\,2})}-\frac{2\sqrt{2}(\bar{r}_2)^3}{(\nu_2 s_2^{\,2})^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_2+3}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$-3\sqrt{2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\frac{\bar{r}_1}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}+3\sqrt{2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\frac{\bar{r}_2}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$-3\sqrt{2}\,\frac{(\bar{r}_1)^2}{\nu_1 s_1^{\,2}}\left\{\nu_1-\frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right\}\frac{\bar{r}_1}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}+3\sqrt{2}\,\frac{(\bar{r}_1)^2}{\nu_1 s_1^{\,2}}\left\{\nu_1-\frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right\}\frac{\bar{r}_2}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma(\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$-3\sqrt{2}\,\frac{(\overline{r}_2)^2}{\nu_2 {s_2}^2}\left\{\nu_2-\frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}\frac{\overline{r}_1}{\sqrt{\nu_1 {s_1}^2}}\,\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}+3\sqrt{2}\,\frac{(\overline{r}_2)^2}{\nu_2 {s_2}^2}\left\{\nu_2-\frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}\frac{\overline{r}_2}{\sqrt{\nu_2 {s_2}^2}}\,\frac{\Gamma(\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$-2\sqrt{2}\,\frac{\left(\overline{r}_1\right)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\,\frac{\Gamma^3\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}+6\sqrt{2}\,\frac{\left(\overline{r}_1\right)^2}{\left(\nu_1 {s_1}^2\right)}\,\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\,\frac{\overline{r}_2}{\sqrt{\nu_2 {s_2}^2}}\,\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}-6\sqrt{2}\,\frac{\overline{r}_1}{\sqrt{\nu_1 {s_1}^2}}\,\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\,\frac{\left(\overline{r}_2\right)^2}{\left(\nu_2 {s_2}^2\right)}\,\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}$$

$$+2\sqrt{2}\,\frac{\left(\overline{r}_2\right)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\,\frac{\Gamma^3\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}$$

$$\therefore m_3=3\sqrt{2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\frac{\overline{r}_1}{\sqrt{\nu_1 {s_1}^2}}\,\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}-3\sqrt{2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\frac{\overline{r}_2}{\sqrt{\nu_2 {s_2}^2}}\,\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}+\frac{2\sqrt{2}(\overline{r}_1)^3}{(\nu_1 {s_1}^2)^{\frac{3}{2}}}\,\frac{\Gamma\left(\frac{\nu_1+3}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}$$

$$-\frac{3\sqrt{2}(\overline{r}_1)^2\nu_1}{(\nu_1 {s_1}^2)}\,\frac{(\overline{r}_2)}{(\nu_2 {s_2}^2)^{\frac{1}{2}}}\,\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}+\frac{3\sqrt{2}(\overline{r}_1)}{(\nu_1 {s_1}^2)^{\frac{1}{2}}}\,\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\,\frac{(\overline{r}_2)^2\nu_2}{(\nu_2 {s_2}^2)}-\frac{2\sqrt{2}(\overline{r}_2)^3}{(\nu_2 {s_2}^2)^{\frac{3}{2}}}\,\frac{\Gamma\left(\frac{\nu_2+3}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$-3\sqrt{2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\frac{\overline{r}_1}{\sqrt{\nu_1 {s_1}^2}}\,\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}+3\sqrt{2}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\frac{\overline{r}_2}{\sqrt{\nu_2 {s_2}^2}}\,\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$-3\sqrt{2}\,\frac{(\overline{r}_1)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\,\nu_1\,\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}+6\sqrt{2}\,\frac{(\overline{r}_1)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\,\frac{\Gamma^3\left(\frac{\nu_1+1}{2}\right)}{\Gamma^3\left(\frac{\nu_1}{2}\right)}+3\sqrt{2}\,\frac{(\overline{r}_1)^2}{\left(\nu_1 {s_1}^2\right)}\,\nu_1\,\frac{\overline{r}_2}{\sqrt{\nu_2 {s_2}^2}}\,\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$+6\sqrt{2}\,\frac{(\overline{r}_1)^2}{\left(\nu_1 {s_1}^2\right)}\,\frac{\overline{r}_2}{\sqrt{\nu_2 {s_2}^2}}\,\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\,\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}-3\sqrt{2}\,\frac{(\overline{r}_2)^2}{\left(\nu_2 {s_2}^2\right)}\,\nu_2\,\frac{\overline{r}_1}{\sqrt{\nu_1 {s_1}^2}}\,\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}$$

$$+6\sqrt{2}\,\frac{(\overline{r}_2)^2}{\left(\nu_2 {s_2}^2\right)}\,\frac{\overline{r}_1}{\sqrt{\nu_1 {s_1}^2}}\,\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\,\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}+3\sqrt{2}\,\frac{\left(\overline{r}_2\right)^3\nu_2}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\,\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}-6\sqrt{2}\,\frac{\left(\overline{r}_2\right)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\,\frac{\Gamma^3\left(\frac{\nu_2+1}{2}\right)}{\Gamma^3\left(\frac{\nu_2}{2}\right)}$$

$$-2\sqrt{2}\,\frac{\left(\overline{r_1}\right)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}+6\sqrt{2}\,\frac{\left(\overline{r_1}\right)^2}{\left(\nu_1 {s_1}^2\right)}\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\frac{\overline{r_2}}{\sqrt{\nu_2 {s_2}^2}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}-6\sqrt{2}\,\frac{\overline{r_1}}{\sqrt{\nu_1 {s_1}^2}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\left(\overline{r_2}\right)^2}{\left(\nu_2 {s_2}^2\right)}\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}$$

$$+2\sqrt{2}\,\frac{\left(\overline{r_2}\right)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_2+1}{2}\right)}{\Gamma^3\left(\frac{\nu_2}{2}\right)}$$

All the other terms cancel out except the $\left(\overline{r_1}\right)^3$ and $\left(\overline{r_2}\right)^3$

Consider only the $\left(\overline{r_1}\right)^3$ terms

$$\therefore m_3 = 2\sqrt{2}\,\frac{\left(\overline{r_1}\right)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_1+3}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}-3\sqrt{2}\,\frac{\left(\overline{r_1}\right)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\nu_1+6\sqrt{2}\,\frac{(\overline{r_1})^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_1+1}{2}\right)}{\Gamma^3\left(\frac{\nu_1}{2}\right)}-2\sqrt{2}\,\frac{\left(\overline{r_1}\right)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_1+1}{2}\right)}{\Gamma^3\left(\frac{\nu_1}{2}\right)}$$

$$-2\sqrt{2}\,\frac{\left(\overline{r_2}\right)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_2+3}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}+3\sqrt{2}\,\frac{\left(\overline{r_2}\right)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\nu_2-6\sqrt{2}\,\frac{(\overline{r_2})^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_2+1}{2}\right)}{\Gamma^3\left(\frac{\nu_2}{2}\right)}+2\sqrt{2}\,\frac{\left(\overline{r_2}\right)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_2+1}{2}\right)}{\Gamma^3\left(\frac{\nu_2}{2}\right)}$$

Now

$\Gamma\left(\frac{\nu_1+3}{2}\right)=\left(\frac{\nu_1+1}{2}\right)\Gamma\left(\frac{\nu_1+1}{2}\right)$ and $\Gamma\left(\frac{\nu_2+3}{2}\right)=\left(\frac{\nu_2+1}{2}\right)\Gamma\left(\frac{\nu_2+1}{2}\right)$

Therefore

$$m_3 = 2\sqrt{2}\,\frac{\left(\overline{r_1}\right)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\left(\frac{\nu_1+1}{2}\right)\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}-3\sqrt{2}\,\frac{\left(\overline{r_1}\right)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\nu_1+6\sqrt{2}\,\frac{(\overline{r_1})^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_1+1}{2}\right)}{\Gamma^3\left(\frac{\nu_1}{2}\right)}-2\sqrt{2}\,\frac{\left(\overline{r_1}\right)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_1+1}{2}\right)}{\Gamma^3\left(\frac{\nu_1}{2}\right)}$$

$$-2\sqrt{2}\,\frac{\left(\overline{r_2}\right)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_2+3}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}+3\sqrt{2}\,\frac{\left(\overline{r_2}\right)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\left(\frac{\nu_2+1}{2}\right)\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\nu_2-6\sqrt{2}\,\frac{(\overline{r_2})^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_2+1}{2}\right)}{\Gamma^3\left(\frac{\nu_2}{2}\right)}+2\sqrt{2}\,\frac{\left(\overline{r_2}\right)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_2+1}{2}\right)}{\Gamma^3\left(\frac{\nu_2}{2}\right)}$$

$$=\sqrt{2}\,\frac{\left(\overline{r_1}\right)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\left\{\nu_1+1-3\nu_1\right\}+4\sqrt{2}\,\frac{(\overline{r_1})^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_1+1}{2}\right)}{\Gamma^3\left(\frac{\nu_1}{2}\right)}$$

$$-\sqrt{2}\,\frac{\left(\bar{r}_2\right)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\left\{\nu_2+1-3\nu_2\right\}-4\sqrt{2}\,\frac{(\bar{r}_2)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_2+1}{2}\right)}{\Gamma^3\left(\frac{\nu_2}{2}\right)}$$

$$=\sqrt{2}\,\frac{\left(\bar{r}_1\right)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\left\{1-2\nu_1\right\}+4\sqrt{2}\,\frac{(\bar{r}_1)^3}{\left(\nu_1 {s_1}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_1+1}{2}\right)}{\Gamma^3\left(\frac{\nu_1}{2}\right)}$$

$$-\sqrt{2}\,\frac{\left(\bar{r}_2\right)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\left\{1-2\nu_2\right\}-4\sqrt{2}\,\frac{(\bar{r}_2)^3}{\left(\nu_2 {s_2}^2\right)^{\frac{3}{2}}}\frac{\Gamma^3\left(\frac{\nu_2+1}{2}\right)}{\Gamma^3\left(\frac{\nu_2}{2}\right)}$$

$$=(\bar{r}_1)^3\left(\frac{2}{\nu_1 {s_1}^2}\right)^{\frac{3}{2}}\frac{\Gamma^3\left(\frac{\nu_1+1}{2}\right)}{\Gamma^3\left(\frac{\nu_1}{2}\right)}\left\{\frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}-\frac{(2\nu_1-1)}{2}\right\}-(\bar{r}_2)^3\left(\frac{2}{\nu_2 {s_2}^2}\right)^{\frac{3}{2}}\frac{\Gamma^3\left(\frac{\nu_2+1}{2}\right)}{\Gamma^3\left(\frac{\nu_2}{2}\right)}\left\{\frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}-\frac{(2\nu_2-1)}{2}\right\}$$

as required.

The fourth moment

$$\mu_4=3(\mu_2)^2=3\left(\frac{1}{n_1}+\frac{1}{n_2}\right)^2$$

$$\mu'_4=\mu_4+4\mu'_1\mu_3+6(\mu'_1)^2\mu_2+(\mu'_1)^4$$

$$\mu'_4=3\left(\frac{1}{n_1}+\frac{1}{n_2}\right)^2+0+6\left((\bar{r}_1)\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 {s_1}^2}}-(\bar{r}_2)\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 {s_2}^2}}\right)^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)$$

$$+\left((\bar{r}_1)\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 {s_1}^2}}-(\bar{r}_2)\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 {s_2}^2}}\right)^4$$

Consider

$$E_{\chi^2_{\nu_1}\chi^2_{\nu_2}}\left((\bar{r}_1)\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 {s_1}^2}}-(\bar{r}_2)\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 {s_2}^2}}\right)^2$$

$$=E_{\chi^2_{\nu_1}\chi^2_{\nu_2}}\left((\bar{r}_1)^2\,\frac{\chi^2_{\nu_1}}{\nu_1 {s_1}^2}-2(\bar{r}_1)\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 {s_1}^2}}(\bar{r}_2)\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 {s_2}^2}}+(\bar{r}_2)^2\,\frac{\chi^2_{\nu_2}}{\nu_2 {s_2}^2}\right)$$

$$=\left((\bar{r}_1)^2\frac{\nu_1}{\nu_1 s_1^{\,2}}-\frac{2(\bar{r}_1)(\bar{r}_2)}{\sqrt{\nu_1 s_1^{\,2}}\sqrt{\nu_2 s_2^{\,2}}}\frac{\sqrt{2}\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\sqrt{2}\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}+(\bar{r}_2)^2\frac{\nu_2}{\nu_2 s_2^{\,2}}\right)$$

$$=(\bar{r}_1)^2\frac{\nu_1}{\nu_1 s_1^{\,2}}+(\bar{r}_2)^2\frac{\nu_2}{\nu_2 s_2^{\,2}}-\frac{4(\bar{r}_1)(\bar{r}_2)}{\sqrt{\nu_1 s_1^{\,2}}\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$E_{\chi^2_{\nu_1}\chi^2_{\nu_2}}\left((\bar{r}_1)\sqrt{\frac{\chi^2_{\nu_1}}{\nu_1 s_1^{\,2}}}-(\bar{r}_2)\sqrt{\frac{\chi^2_{\nu_2}}{\nu_2 s_2^{\,2}}}\right)^4=$$

$$E_{\chi^2_{\nu_1}\chi^2_{\nu_2}}\left\{(\bar{r}_1)^4\frac{(\chi^2_{\nu_1})^2}{(\nu_1 s_1^{\,2})^2}-4\frac{(\bar{r}_1)^3(\bar{r}_2)(\chi^2_{\nu_1})^{\frac{3}{2}}(\chi^2_{\nu_2})^{\frac{1}{2}}}{(\nu_1 s_1^{\,2})^{\frac{3}{2}}(\nu_2 s_2^{\,2})^{\frac{1}{2}}}+\frac{6(\bar{r}_1)^2(\bar{r}_2)^2\chi^2_{\nu_1}\chi^2_{\nu_2}}{(\nu_1 s_1^{\,2})(\nu_2 s_2^{\,2})}\right.$$

$$\left.-4\frac{(\bar{r}_1)(\bar{r}_2)^3(\chi^2_{\nu_1})^{\frac{1}{2}}(\chi^2_{\nu_2})^{\frac{3}{2}}}{(\nu_1 s_1^{\,2})^{\frac{1}{2}}(\nu_2 s_2^{\,2})^{\frac{3}{2}}}+(\bar{r}_2)^4\frac{(\chi^2_{\nu_2})^2}{(\nu_2 s_2^{\,2})^2}\right\}$$

$$=\left\{(\bar{r}_1)^4\frac{\nu_1(\nu_1+2)}{(\nu_1 s_1^{\,2})^2}-\frac{8(\bar{r}_1)^3(\bar{r}_2)}{(\nu_1 s_1^{\,2})^{\frac{3}{2}}(\nu_2 s_2^{\,2})^{\frac{1}{2}}}\frac{(\nu_1+2)\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}+\frac{6(\bar{r}_1)^2(\bar{r}_2)^2\nu_1\nu_2}{(\nu_1 s_1^{\,2})(\nu_2 s_2^{\,2})}\right.$$

$$\left.-\frac{8(\bar{r}_1)(\bar{r}_2)^3}{(\nu_1 s_1^{\,2})^{\frac{1}{2}}(\nu_2 s_2^{\,2})^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{(\nu_2+2)\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}+(\bar{r}_2)^4\frac{\nu_2(\nu_2+2)}{(\nu_2 s_2^{\,2})^2}\right\}$$

Fourth moment about zero unconditional

$$m'_4=3\left(\frac{1}{n_1}+\frac{1}{n_2}\right)^2+6\left((\bar{r}_1)^2\frac{\nu_1}{\nu_1 s_1^{\,2}}+(\bar{r}_2)^2\frac{\nu_2}{\nu_2 s_2^{\,2}}-\frac{4(\bar{r}_1)(\bar{r}_2)}{\sqrt{\nu_1 s_1^{\,2}}\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)$$

$$\left\{(\bar{r}_1)^4\frac{\nu_1(\nu_1+2)}{(\nu_1 s_1^{\,2})^2}-\frac{8(\bar{r}_1)^3(\bar{r}_2)}{(\nu_1 s_1^{\,2})^{\frac{3}{2}}(\nu_2 s_2^{\,2})^{\frac{1}{2}}}\frac{(\nu_1+2)\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}+\frac{6(\bar{r}_1)^2(\bar{r}_2)^2\nu_1\nu_2}{(\nu_1 s_1^{\,2})(\nu_2 s_2^{\,2})}\right.$$

$$\left.-\frac{8(\bar{r}_1)(\bar{r}_2)^3}{(\nu_1 s_1^{\,2})^{\frac{1}{2}}(\nu_2 s_2^{\,2})^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{(\nu_2+2)\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}+(\bar{r}_2)^4\frac{\nu_2(\nu_2+2)}{(\nu_2 s_2^{\,2})^2}\right\}.$$

Fourth moment about the mean unconditional

$$m_4 \quad = m_4^{'} - 4m_1^{'}m_3 - 6\left(m_1^{'}\right)^2 m_2 - \left(m_1^{'}\right)^4$$

$$\therefore -4m_1^{'}m_3 = -4\sqrt{2}\left(\frac{(\bar{r}_1)}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)} - \frac{(\bar{r}_2)}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)\times$$

$$\left\{\frac{(\bar{r}_1)^3 2^{\frac{3}{2}}}{(\nu_1 s_1^{\,2})^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_1+3}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)} - \frac{3\sqrt{2}(\bar{r}_1)^2\nu_1}{(\nu_1 s_1^{\,2})}\frac{(\bar{r}_2)}{(\nu_2 s_2^{\,2})^{\frac{1}{2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)} + \frac{3\sqrt{2}(\bar{r}_1)}{(\nu_1 s_1^{\,2})^{\frac{1}{2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{(\bar{r}_2)^2\nu_2}{(\nu_2 s_2^{\,2})} - \frac{(\bar{r}_2)^3}{(\nu_2 s_2^{\,2})^{\frac{3}{2}}}\frac{2^{\frac{3}{2}}\Gamma\left(\frac{\nu_2+3}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right.$$

$$-3\sqrt{2}\left(\frac{(\bar{r}_1)^2}{\nu_1 s_1^{\,2}}\left(\frac{\bar{r}_1}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)} - \frac{\bar{r}_2}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma(\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)\left\{\nu_1 - \frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right\}\right)$$

$$-3\sqrt{2}\,\frac{(\bar{r}_2)^2}{\nu_2 s_2^{\,2}}\left(\frac{\bar{r}_1}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)} - \frac{\bar{r}_2}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma(\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)\left\{\nu_2 - \frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}$$

$$\left. -2\frac{\left(\bar{r}_1\right)^2}{\nu_1 s_1^{\,2}}\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)} - 2\frac{\left(\bar{r}_2\right)^2}{\nu_2 s_2^{\,2}}\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)} + 4\frac{\bar{r}_1}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\bar{r}_2}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right\}$$

and

$$-6\left(m_1^{'}\right)^2 m_2 \quad = -12\left(\frac{(\bar{r}_1)}{\sqrt{\nu_1 s_1^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)} - \frac{(\bar{r}_2)}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)^2\times$$

$$\left[\left(\frac{1}{n_1}+\frac{1}{n_2}\right) + \left(\frac{(\bar{r}_1)^2}{\nu_1 s_1^{\,2}}\left\{\nu_1 - \frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right\} + \frac{(\bar{r}_2)^2}{\nu_2 s_2^{\,2}}\left\{\nu_2 - \frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}\right)\right]$$

$$= -12\left\{\frac{(\bar{r}_1)^2}{\nu_1 s_1^{\,2}}\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)} + \frac{(\bar{r}_2)^2}{\nu_2 s_2^{\,2}}\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)} - \frac{2(\bar{r}_1)}{\sqrt{\nu_1 s_1^{\,2}}}\frac{(\bar{r}_2)}{\sqrt{\nu_2 s_2^{\,2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right\}\times$$

$$\left[\left(\frac{1}{n_1}+\frac{1}{n_2}\right)+\left(\frac{(\bar{r}_1)^2}{\nu_1 {s_1}^2}\left\{\nu_1-\frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right\}+\frac{(\bar{r}_2)^2}{\nu_2 {s_2}^2}\left\{\nu_2-\frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}\right)\right]$$

$$-\left(m_1^{'}\right)^4=-4\left(\frac{(\bar{r}_1)}{\sqrt{\nu_1 {s_1}^2}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}-\frac{(\bar{r}_2)}{\sqrt{\nu_2 {s_2}^2}}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right)^4$$

$$=-4\left\{\frac{(\bar{r}_1)^4}{\left(\nu_1 {s_1}^2\right)^2}\frac{\Gamma^4\left(\frac{\nu_1+1}{2}\right)}{\Gamma^4\left(\frac{\nu_1}{2}\right)}+4\frac{(\bar{r}_1)^3(\bar{r}_2)}{(\nu_1 {s_1}^2)^{\frac{3}{2}}(\nu_2 {s_2}^2)^{\frac{1}{2}}}\frac{\Gamma^3\left(\frac{\nu_1+1}{2}\right)}{\Gamma^3\left(\frac{\nu_1}{2}\right)}\frac{\Gamma\left(\frac{\nu_2+1}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\right.$$

$$\left.+6\frac{(\bar{r}_1)^2(\bar{r}_2)^2}{(\nu_1 {s_1}^2)(\nu_2 {s_2}^2)}\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}-4\frac{(\bar{r}_1)(\bar{r}_2)^3}{(\nu_1 {s_1}^2)^{\frac{1}{2}}(\nu_2 {s_2}^2)^{\frac{3}{2}}}\frac{\Gamma\left(\frac{\nu_1+1}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\Gamma^3\left(\frac{\nu_2+1}{2}\right)}{\Gamma^3\left(\frac{\nu_2}{2}\right)}+\frac{(\bar{r}_2)^4}{\left(\nu_2 {s_2}^2\right)^2}\frac{\Gamma^4\left(\frac{\nu_2+1}{2}\right)}{\Gamma^4\left(\frac{\nu_2}{2}\right)}\right\}$$

$$m_4=3\left(\frac{1}{n_1}+\frac{1}{n_2}\right)^2+6(\bar{r}_1)^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\frac{1}{(\nu_1 {s_1}^2)}\left\{\nu_1-\frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right\}$$

$$+6(\bar{r}_2)^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\frac{1}{(\nu_2 {s_2}^2)}\left\{\nu_2-\frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}$$

$$+\frac{3(\bar{r}_1)^4}{(\nu_1 {s_1}^2)^2}\left\{\frac{\nu_1(\nu_1+2)}{3}+4\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\left[\frac{1}{3}\left(\nu_1-2\right)-\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\right]\right\}$$

$$+\frac{3(\bar{r}_2)^4}{(\nu_2 {s_2}^2)^2}\left\{\frac{\nu_2(\nu_2+2)}{3}+4\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\left[\frac{1}{3}\left(\nu_2-2\right)-\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right]\right\}$$

$$+\frac{6(\bar{r}_1)^2(\bar{r}_2)^2}{(\nu_1 {s_1}^2)(\nu_2 {s_2}^2)}\left\{\nu_1\nu_2+4\frac{\Gamma^2\left(\frac{\nu_1+1}{2}\right)}{\Gamma^2\left(\frac{\nu_1}{2}\right)}\frac{\Gamma^2\left(\frac{\nu_2+1}{2}\right)}{\Gamma^2\left(\frac{\nu_2}{2}\right)}-\frac{2\Gamma^2\left(\frac{\nu_1+1}{2}\right)\nu_2}{\Gamma^2\left(\frac{\nu_1}{2}\right)}-\frac{2\Gamma^2\left(\frac{\nu_2+1}{2}\right)\nu_1}{\Gamma^2\left(\frac{\nu_2}{2}\right)}\right\}$$

REFERENCES

1. Berger, J.O. and Sun, D. (2008). Objective priors for the Bivariate normal model. *The Annals of Statistics* 36(2), 963-982.
2. Bechhofer, R.E. (1954). A single-sample multiple decision procedure for ranking means of normal populations with known variances. *The Annals of Mathematical Statistics* 25, 16–39.
3. Chow, You-Min and Owen, D.B. (1989). On the distributions of the estimated process capability indices. *Commun. Statist-Theory Meth.* 18(12) 4549-4560.
4. Cornish, E.A. and Fisher, R.A. (1937). Moments and cumulants in the specification of distributions. *Revue de l'Institut International de Statistisque,* 5, 307-320.
5. Elderton, W.P (1953). *Frequency curves and correlation*. Cambridge University Press.
6. Elderton, W.P. and Johnson, N.L. (1969). *Systems of Frequency Curves*. Cambridge University Press.
7. Fisher, R.A. and Cornish, E.A. (1960). The Percentile Points of distributions having known cumulants. *Technometrics*, 2, 209–226.
8. Gupta S.S. (1956). On a decision rule for a problem in ranking means. Mimeo. Ser. No. 150. Inst. Of Statistics, University of North Carolina, Chapel Hill, North Carolina.
9. Hoshino, T. (2008). Bayesian significance testing and multiple comparison from MCMC outputs. *Computational Statistical & Data Analysis,* 52, 3543-3559.
10. Jeffreys, H. (1961). *Theory of Probabaility.* Oxford University Press, New York, N.Y.
11. Reilly, F.K and Brown, K.C. (2003*), Investment Analysis and Portfolio Management. 8th edition.* Thomson South Western.
12. Zellner, A. (1971). *An Introduction to Bayesian Inference in Econometrics.* New York. John Wiley & Sons.

13. Zigler, B. (2009). How much Oil can Gold buy? *HardAssetsInvestor.com* (HAI) Accessed on the web: 13 November 2009.*www.hardassetsinvestor.com/.../1538-how-**much**-**oil**-**can**-**gold**-**buy**.html?*
14. Zigler, B. (2009). Gold or Oil: What's a better inflation hedge? *HardAssetsInvestor.com* (HAI), Accessed on the web:13 November 2009. *www.hardassetsinvestor.com/.../1722-**gold-or-oil**-**whats-a-better**-**inflation**-**hedge**.html -*