Improved regression-type estimation of the index of a stable distribution using the characteristic function

J. Martin van Zyl

Abstract Refinements with respect to heteroscedasticity of the residuals is made to the regression estimation method, based on the sample characteristic function, to estimate the index and scale parameter of the stable distribution. The refined procedure leads to estimates where the bias is decreased significantly.

Keyword: *Stable Distribution, Index, Characteristic function, Estimation*

1 Introduction

The purpose of this work is to make a refinement to the regression estimation method based on the characteristic function proposed by Koutrouvellis (1980). The procedure derived in his work leads to a regression model with heteroscedasticity of the regression error terms and this aspect is improved. By designing the model in such a way that the variance of the error terms is a minimum, a significant decrease in the bias of the estimated parameters was also found.

The characteristic function $\phi(t)$ of the stable distribution is given by

$$\log\phi(t) = -\sigma^{\alpha} |t|^{\alpha} \{1 - i\beta sign(t)\tan(\pi\alpha/2)\} + i\mu t,\ \alpha \neq 1,$$

and $$\log\phi(t) = -\sigma^{\alpha} |t|^{\alpha} \{1 - i\beta sign(t)\log(|t|)\} + i\mu t,\ \alpha = 1.$$

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The parameters are the index $\alpha \in (0,2]$, scale parameter $\sigma > 0$, coefficient of skewness $\beta \in [-1,1]$ and mode μ. The symmetric case with $\mu = 0, \beta = 0$ will be considered in this work. The notation and review is based on the work of Weron, p915 in the book by (Gentle, Härdle, Mori, eds , 2004). Koutrouvellis (1980) made use of the properties of the characteristic function and using the fact that $|\phi(t)|^2 = \exp(-2\sigma^\alpha |t|^\alpha)$ derived the model

$$\log(-\log(|\phi(t)|^2)) = \log(2\sigma^\alpha) + \alpha \log(|t|), \qquad (1.1)$$

a simple linear regression model can be formed

$$y_k = m + \alpha \omega_k + \varepsilon_k. \qquad (1.2)$$

The characteristic function is estimated for a given value of t, for a sample of size n i.i.d. observations $x_1,...,x_n$, as $\hat{\phi}_n(t) = \frac{1}{n}\sum_{j=1}^{n} e^{itx_j}$, and $y_k = \log(-\log(|\hat{\phi}_n(t_k)|^2))$, $m = \log(2\sigma^\alpha)$, $\omega_k = \log(|t_k|)$, ε_k an error term. Koutrouvellis (1980) suggested using $t_k = \pi k / 25, k = 1,..., K$, and optimal values of K was suggested for various sample sizes and $\alpha's$.

An expression for the covariance $\mathrm{cov}(|\hat{\phi}_n(t_j)|^2, |\hat{\phi}_n(t_k)|^2)$ and thus the variance of $|\hat{\phi}_n(t_j)|^2$ is given by Koutrouvellis (1980). This expression is quite complicated and depends on the unknown parameters, and thus also $var(\log(-\log(|\hat{\phi}_n(t_j)|^2)))$ making weighted regression problematic.

Much research was done on using an approximate covariance matrix and generalized least squares to estimate the parameters. Feuerberger and McDunnough (1981) considered the asymptotic joint distribution of $\phi(t_1),...,\phi(t_K)$. An excellent overview of this approach is given in the paper by Besbeas and Morgan (2008). They suggested arithmetic spacing instead of choosing uniformly spaced t's over an interval and this approach yielded good results.

It is shown that the residual variance is highly heteroscedastistic with respect to t, and that the t's suggested by Koutrouvellis (1980), is in most case over the interval where the residual variance is changing most as a function of the t's. This might lead to a decrease in the efficiency, and also incorrect estimates of the variances of the estimated parameters.

An interval where the variance of the residuals is almost constant and small is suggested in this work. The variance of the residuals, $\varepsilon 's$ for a given t and the true parameters, is estimated using simulated samples. Residuals for a sample was calculated using the true parameters as $\varepsilon_k = y_k - m - \alpha\omega_k$, and from these residuals the variance, $var(\varepsilon_k) = var(\varepsilon_k \mid t, \sigma, \alpha)$, was estimated.

The simulation study suggests approximate constant and minimum residual variances for $t \in [0.5,1]$. With K equally spaced t's in these intervals (Koutrouvellis 1980). Performing the regression using K values of t chosen in this interval, resulted in a much smaller bias of the estimated parameters.

2. Estimation of the residual variance

Using the true parameters and simulated samples, residuals for a given t and a simulated sample was calculated using equation (1.2). The variance of the residuals of $M = 1000$ samples of size $n = 200$ each, with respect to t is shown. The data was simulated with $\alpha = 0.7, 1.5$, $\sigma = 0.1, 1.0, 2.0$ and $\beta = 0, \mu = 0$.

Approximate constant residual variance was found in two intervals, the one for t between 0.5 and 1.0, the other for t larger than a certain point in the region of 2 - 4, depending on the values of α and σ. The error variance is smallest in the interval 0.5 to 1.0. Note that heteroscedasticity would still be present in these intervals.

The covariance matrices of the estimated parameters will be given. These are calculated from the M=1000 estimated parameters using the regression method on the intervals $t \in [0.5, 1.0], t \in [3.0, 5.0]$ respectively. K denote the optimal number of observations as suggested by Koutrouvellis (1980), and K uniformly chosen t's will be chosen over in the intervals $t \in [0.5, 1.0], t \in [3.0, 5.0]$ to estimate the parameters. The estimated covariance matrices of the estimated parameters on the two intervals respectively, $\hat{m}$ and $\hat{\alpha}$, will be denoted by $\hat{\Phi}_1$ and $\hat{\Phi}_2$,

$(\hat{\Phi})_{11} = \mathrm{var}(\hat{m}), (\hat{\Phi})_{22} = \mathrm{var}(\hat{\alpha}), (\hat{\Phi})_{12} = (\hat{\Phi})_{21} = \mathrm{cov}(\hat{\alpha}, \hat{m})$.

The true parameters are used to calculate a residual from the simulated observations, for example $\hat{\sigma}_t^2 = \hat{\sigma}_t^2 \mid t, \sigma, \alpha = \mathrm{cov}(\varepsilon_1, ..., \varepsilon_M \mid t, \alpha, \sigma),$, where

$\varepsilon_j \mid t = y_j - m - \alpha \log(t)$, $m = \log(2\sigma^{\alpha}), y_j = \log(-\log(|\hat{\phi}_n(t)|^2)), j = 1, ..., M$.

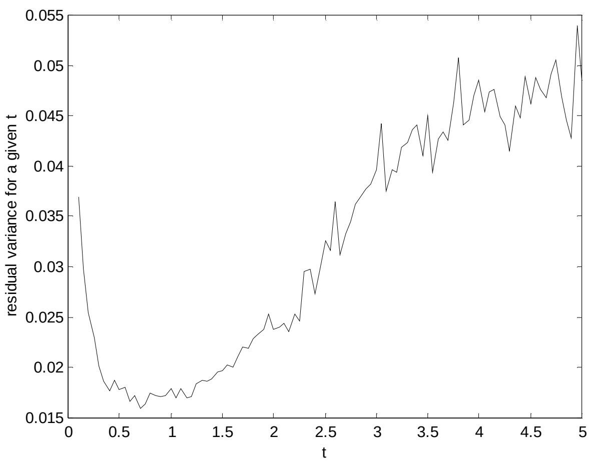

Fig. 1 Estimated residual variances for given values of t, $\alpha = 0.7, \sigma = 0.1, n = 200$.

$$\hat{\Phi}_1 = \begin{pmatrix} 0.0360 & 0.0148 \\ 0.0148 & 0.0726 \end{pmatrix} \qquad \hat{\Phi}_2 = \begin{pmatrix} 0.1439 & -0.0926 \\ -0.0926 & 0.0668 \end{pmatrix}.$$

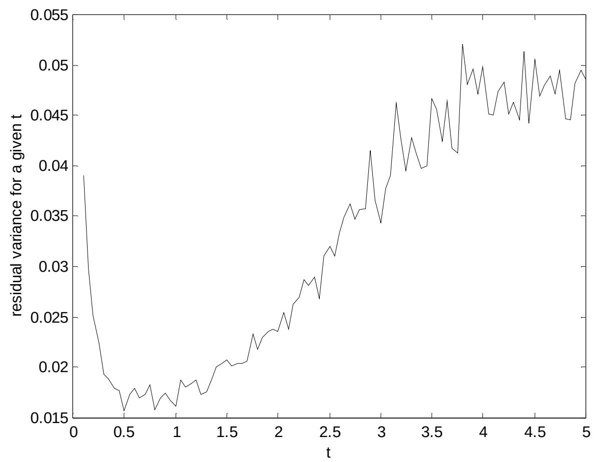

Fig. 2 Estimated residual variances for given values of t, $\alpha = 0.7, \sigma = 1.0, n = 200.$

$$\hat{\Phi}_1 = \begin{pmatrix} 0.0181 & 0.0145 \\ 0.0145 & 0.0423 \end{pmatrix} \qquad \hat{\Phi}_2 = \begin{pmatrix} 0.5684 & -0.3986 \\ -0.3986 & 0.2900 \end{pmatrix}.$$

For the cases considered in figures 1 and 2 with $\alpha = 0.7$, Koutrouvellis (1980) the optimal K was 30, and the suggested t's are between 0.1257 and 3.7699, an interval where the variances are heteroscedastistic. The estimation of m was stable, but σ estimated using the second interval was unstable and many extremely large values occurred in both cases $\sigma = 0.1, 1.0$.

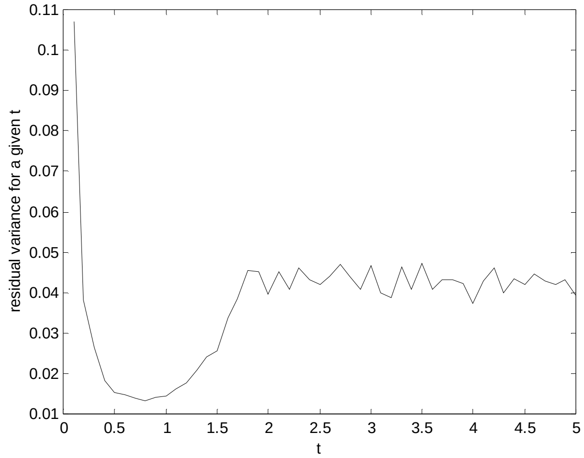

Fig. 3 Estimated residual variances for given values of t, $\alpha = 1.5, \sigma = 0.1, n = 200.$

$$\hat{\Phi}_1 = \begin{pmatrix} 0.1042 & -0.0229 \\ -0.0229 & 0.2163 \end{pmatrix} \qquad \hat{\Phi}_2 = \begin{pmatrix} 0.0002 & 0.0020 \\ 0.0020 & 0.0396 \end{pmatrix}.$$

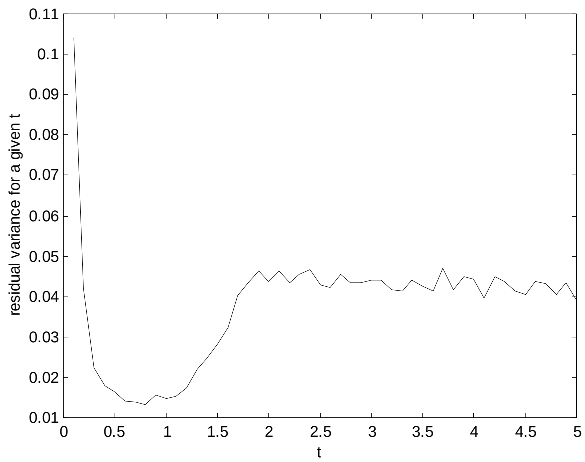

Fig. 4 Estimated residual variances for given values of t, $\alpha = 1.5, \sigma = 1.0, n = 200.$

$$\hat{\Phi}_1 = \begin{pmatrix} 0.0151 & 0.0084 \\ 0.0084 & 0.0269 \end{pmatrix} \qquad \hat{\Phi}_2 = \begin{pmatrix} 0.8192 & -0.5987 \\ -0.5987 & 0.4469 \end{pmatrix}.$$

For the cases considered in figures 3 and 4, Koutrouvellis (1980) suggested using K=11 and the suggested t's are between 0.1257 and 1.3823, an interval where the variances are highly heteroscedastistic. The estimation of σ was unstable using the interval 3.0 to 5.0 for $\sigma = 1.0, 2.0$ and many extremely large estimated values occurred. The estimated parameters using the second interval was extremely biased.

In figure 5 the estimated residual variance for t given is plotted against log(t).

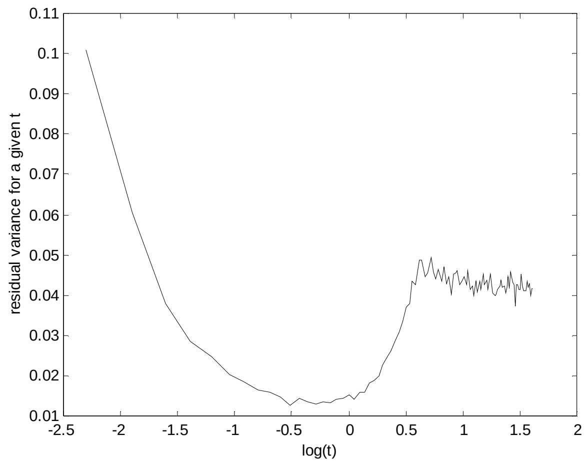

Fig. 5 Estimated residual variances plotted against log(t), $\alpha = 1.5, \sigma = 1.0, n = 200$.

From the above figures it can be seen that the residual variances reaches a minimum and are reasonably constant or homoscedastic in the interval $t \in [0.5, 1.0]$.

3 Comparison between estimation procedures

In this section a simulation study was conducted to compare the performance of estimation using the interval 0.5 and 1.0 with estimation when choosing $t_k = \pi k / 25, k = 1, ..., K$. Three values of σ was investigated, $\sigma = 0.1, 1.0, 2.0$. The numbers of t's was chosen as K, as suggested by Koutrouvellis (1980). Thus for example the interval $t \in [0.5, 1]$ will mean that $t_k = 0.5 + (0.5 / K)k, k = 0, ..., K - 1$. Samples from symmetric distributions with location parameter zero were considered. The results are based on 5000 simulated samples each time. The MSE and bias was calculated with respect to the true parameter. For very small values of α , negative estimates may occur, and in such a case it was adjusted to 0.3. This happened in a very small proportion of cases, less than one in a hundred when $\alpha = 0.5$. No adjustment was made if the estimate was larger than 2, which also occurred in a small proportion for α close to 2.

	N=200, $\hat{\alpha}$					
$\sigma = 0.1, \beta = 0, \mu = 0$	0.5-1.0 interval			Koutrouvelis		
α	Mean	Bias	MSE	Mean	Bias	MSE
1.9 (K=9)	1.9535	-0.0535	0.0421	1.9685	-0.0685	0.0257
1.7 (K=10)	1.8027	-0.1027	0.1610	1.8549	-0.1549	0.1051
1.5 (K=11)	1.6167	-0.1167	0.2287	1.6839	-0.1839	0.1611
1.3 (K=22)	1.3838	-0.0838	0.2273	1.4138	-0.1130	0.0889
1.1 (K=24)	1.1523	-0.0523	0.1650	1.1731	-0.0731	0.0537
0.9 (K=28)	0.9225	-0.0225	0.1120	0.9289	-0.0289	0.0218
0.7 (K=30)	0.7225	-0.0225	0.0734	0.7132	-0.0132	0.0102
0.5 (K=86)	0.5331	-0.0331	0.0391	0.5048	-0.0048	0.0030

Table 1 Comparison of estimation procedures of α with respect to bias and MSE, $\sigma = 0.1, n = 200$.

	n=200, $\hat{\alpha}$					
$\sigma = 1, \beta = 0, \mu = 0$	0.5 – 1.0 interval			Koutrouvelis		
α	Mean	Bias	MSE	Mean	Bias	MSE
1.9 (K=9)	1.9021	-0.0021	0.0094	1.9064	-0.0064	0.0068
1.7 (K=10)	1.6988	0.0012	0.0191	1.7077	-0.0077	0.0127
1.5 (K=11)	1.5011	-0.0011	0.0261	1.5080	-0.0080	0.0130
1.3 (K=22)	1.2956	0.0044	0.0330	1.2473	0.0527	0.0131
1.1 (K=24)	1.0998	0.0002	0.0386	1.0668	0.0340	0.0099
0.9 (K=28)	0.9007	-0.0007	0.0432	0.8742	0.0258	0.0075
0.7 (K=30)	0.7088	-0.0088	0.0423	0.6915	0.0085	0.0057
0.5 (K=86)	0.5151	-0.0151	0.0310	0.4584	0.0416	0.0043

Table 2 Comparison of estimation procedures of α with respect to bias and MSE, $\sigma = 1.0, n = 200$.

	n=800, $\hat{\alpha}$					
$\sigma = 2, \beta = 0, \mu = 0$	0.5 – 1.0 interval			Koutrouvelis		
α	Mean	Bias	MSE	Mean	Bias	MSE
1.9 (K=9)	1.8474	0.0526	0.0614	1.8282	0.0719	0.0108
1.7 (K=10)	1.6782	0.0218	0.0474	1.6263	0.0737	0.0113
1.5 (K=11)	1.4917	0.0083	0.0362	1.4381	0.0619	0.0093
1.3 (K=16)	1.2936	0.0064	0.0276	1.1481	0.1519	0.0268
1.1 (K=18)	1.0956	0.0044	0.0214	0.9997	0.1003	0.0134
0.9 (K=22)	0.8977	0.0023	0.0183	0.8358	0.0650	0.0069
0.7 (K=24)	0.6988	0.0012	0.0161	0.6623	0.0377	0.0036
0.5 (K=68)	0.4993	0.0007	0.0118	0.4607	0.0393	0.0026

Table 3 Comparison of estimation procedures of α with respect to bias and MSE, $\sigma = 2.0, n = 200$.

In the following three tables, the estimated σ's using the two procedures are compared.

	n=200, $\hat{\sigma}$					
$\sigma = 0.1, \beta = 0, \mu = 0$	0.5 -1.0 interval			Koutrouvelis		
α	Mean	Bias	MSE	Mean	Bias	MSE
1.9 (K=9)	0.1063	-0.0063	0.0003	0.1079	-0.0079	0.0003
1.7 (K=10)	0.1163	-0.0163	0.0017	0.1210	-0.0210	0.0015
1.5 (K=11)	0.1220	-0.0220	0.0037	0.1272	-0.0272	0.0030
1.3 (K=22)	0.1202	-0.0202	0.0055	0.1127	-0.0127	0.0014
1.1 (K=24)	0.1173	-0.0173	0.0062	0.1091	-0.0091	0.0011
0.9 (K=28)	0.1144	-0.0144	0.0069	0.1039	-0.0039	0.0008
0.7 (K=30)	0.1176	-0.0176	0.0084	0.1024	-0.0024	0.0008
0.5 (K=86)	0.1249	-0.0249	0.0105	0.1012	-0.0012	0.0005

Table 4 Comparison of estimation procedures of σ with respect to bias and MSE, $\sigma = 0.1, n = 200$.

	n=200, $\hat{\sigma}$					
$\sigma = 1, \beta = 0, \mu = 0$	0.5 – 1.0 interval			Koutrouvelis		
α	Mean	Bias	MSE	Mean	Bias	MSE
1.9 (K=9)	0.9957	0.0043	0.0036	0.9964	0.0036	0.0035
1.7 (K=10)	0.9959	0.0041	0.0050	0.9973	0.0027	0.0046
1.5 (K=11)	0.9955	0.0045	0.0069	0.9965	0.0035	0.0060
1.3 (K=22)	0.9901	0.0099	0.0096	0.9608	0.0392	0.0080
1.1 (K=24)	0.9884	0.0116	0.0152	0.9716	0.0284	0.0092
0.9 (K=28)	0.9834	0.0166	0.0235	0.9757	0.0243	0.0116
0.7 (K=30)	0.9847	0.0153	0.0405	0.9908	0.0092	0.0175
0.5 (K=86)	0.9795	0.0205	0.0689	1.0373	-0.0373	0.0538

Table 5 Comparison of estimation procedures of σ with respect to bias and MSE, $\sigma = 1.0, n = 200$.

	n=800, $\hat{\sigma}$					
$\sigma = 2, \beta = 0, \mu = 0$	0.5 – 1.0 interval			Koutrouvelis		
α	Mean	Bias	MSE	Mean	Bias	MSE
1.9 (K=9)	2.0102	-0.0102	0.0062	1.9497	0.0503	0.0083
1.7 (K=10)	2.0053	-0.0053	0.0069	1.9484	0.0516	0.0090
1.5 (K=11)	2.0057	-0.0057	0.0091	1.9559	0.0441	0.0091
1.3 (K=16)	2.0067	-0.0067	0.0125	1.9271	0.0729	0.0138
1.1 (K=18)	2.0083	-0.0083	0.0183	1.9590	0.0410	0.0120
0.9 (K=22)	2.0134	-0.0134	0.0300	0.8350	0.0650	0.0069
0.7 (K=24)	2.0278	-0.0278	0.0534	2.0230	-0.0230	0.0244
0.5 (K=68)	2.0442	-0.0442	0.0951	2.1632	-0.1632	0.1112

Table 6 Comparison of estimation procedures of σ with respect to bias and MSE, $\sigma = 2.0, n = 800$.

It can be seen that for especially larger values of the index α, the suggested interval 0.5 to 1.0 for t, leads to much better results with respect to bias and also MSE. For $\alpha < 1$ the Koutrouvellis (1980) procedure is stable and might be better to use.

It was found that the estimated variance of the slope using the usual regression estimator lead to totally incorrect estimated variances of the estimated parameters, showing that there is still much heteroscedasticity in the interval 0.5 to 1.0. Bootstrap estimates of the variance of the slope (α) yielded much better estimates for a single sample.

4 Conclusions

The interval proposed outperforms those derived by Koutrouvellis (1980) with respect to bias. Refinements can be made to this work, with respect to the number of t's used in the regression. This method is much simpler than those using an approximate covariance matrix. The simulation study shows that the method performs well with respect to bias and MSE over the whole range of parameters commonly encountered in practical problems.

The ideal would be to not only lower the bias but also the MSE and research using principles of optimal experimental design applied further can maybe lead to such an estimation technique.

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