Bayesian process control for the p - chart

Lizanne Raubenheimer[1] and Abrie J. van der Merwe[2]

[1]Department of Statistics, Rhodes University

[2]Department of Mathematical Statistics and Actuarial Science, University of the Free State

Abstract: The binomial distribution is often used in quality control. The usual operation of the p - chart will be extended by introducing a Bayesian approach. We will consider a beta prior, six different ones. Control chart limits, average run lengths and false alarm rates will be determined by using a Bayesian method. These results will be compared to the results obtained when using the classical method. A predictive density based on a Bayesian approach will be used to derive the rejection region. The proposed method gives wider control limits than those obtained from the classical method. The Bayesian method gives larger values for the average run length and smaller values for the false alarm rate. A smaller value for the false alarm rate is desired.

Key words: Average run length, Bayesian analysis, Beta-binomial distribution, False alarm rates, p - chart, Predictive density

1 Introduction

Quality control is a process which is used to maintain the standards of products produced or services delivered. The binomial distribution is often used in quality control. The proportion, p, denotes the proportion of defective items in the population. In this paper the focus will be on the control chart for the proportion of nonconforming or defective products produced by a process, i.e. the p - chart.

Control chart limits, average run lengths and false alarm rates will be determined by using a Bayesian method. These results will be compared to the results obtained when using the classical (frequentist) method. Calabrese (1995) states that attributes control techniques, such as p - charts, plot statistics related to defective items and call for corrective action if the number of defectives becomes too large. The goal is to decide on the basis of the sample data whether the production process has shifted from an in-control state to an out-of-control state. If the production process shifted to an out-of-control state, the process should be inspected and repaired. Chakraborti & Human (2006) examined the effects of parameter estimation for the p - chart. Calabrese (1995) considered a Bayesian process control procedure with fixed samples sizes and sampling intervals where the proportion of defectives is the quality variable of interest. Hamada (2002) considered Bayesian tolerance interval control limits for attributes. Menzefricke (2002) proposed a Bayesian approach to obtain control charts when there is parameter uncertainty, using a predictive distribution to derive the rejection region. Menzefricke (2002) assumed that the prior information on p, the proportion of defective items in the population, is a beta distribution which means that the posterior distribution of p will also be a beta distribution.

Let X_i follow a binomial distribution with parameters n and p. If the value for p is unknown, p should be estimated from the observed sample data. The proportion of nonconforming items from sample i is defined as

$$\hat{p}_i = \frac{X_i}{n} \qquad i = 1, 2, \ldots, m$$

and then $\overline{p}$ is calculated, where $\overline{p}$ is the average of the sample proportions and is defined as

$$\overline{p} = \frac{\sum_{i=1}^{m} \hat{p}_i}{m},$$

where n is the size of each sample, and m is the number of samples. From Montgomery (1996) the frequentist control chart is defined as:

$$\begin{aligned} \text{UCL} &= \overline{p} + 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}} \\ \text{Centre line} &= \overline{p} \\ \text{LCL} &= \overline{p} - 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}. \end{aligned}$$

Chakraborti & Human (2006) state that if p is unknown, the common practice is to estimate the proportion in phase I of the study when the process is thought to be in-control. The size of each sample, n, is assumed to be equal, which is not always the case in practice. We will introduce a Bayesian approach for the p - chart. For the Bayesian method, the predictive density will be used to determine the control chart. From a Bayesian point, we have to decide on a prior for this unknown value of p. Jensen *et al.* (2006) states the following: Updating control chart limits would fit naturally in a Bayesian control chart scheme (Hamada (2002)) where prior estimates are updated resulting in posterior estimates that can continue to be updated over the life of the monitoring scheme. The Bayesian method will be discussed in Section 2, a simulation study will be given in Section 3 and an example will be considered in Section 4. Concluding remarks will be discussed in Section 5.

2 Bayesian Method

Menzefricke (2002) proposed a Bayesian approach to obtain control charts when there is parameter uncertainty, using a predictive distribution to derive the rejection region. Menzefricke (2002) assumed that the prior information on p, the proportion of defective items in the population, is a beta distribution which means that the posterior distribution of p will also be a beta distribution. The beta prior is a conjugate prior to the binomial distribution. Consider a beta prior, i.e. $p \sim Beta(a,b)$ for the unknown p

$$\pi(p) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1-p)^{b-1}. \tag{1}$$

Figure 1 shows different plots of the beta prior for a number of values for a and b. We considered six different priors, for illustration purposes.

For the p - chart the likelihood follows as

$$L(p|data) \propto p^{\sum_{i=1}^{m} x_i}(1-p)^{mn-\sum_{i=1}^{m} x_i}. \tag{2}$$

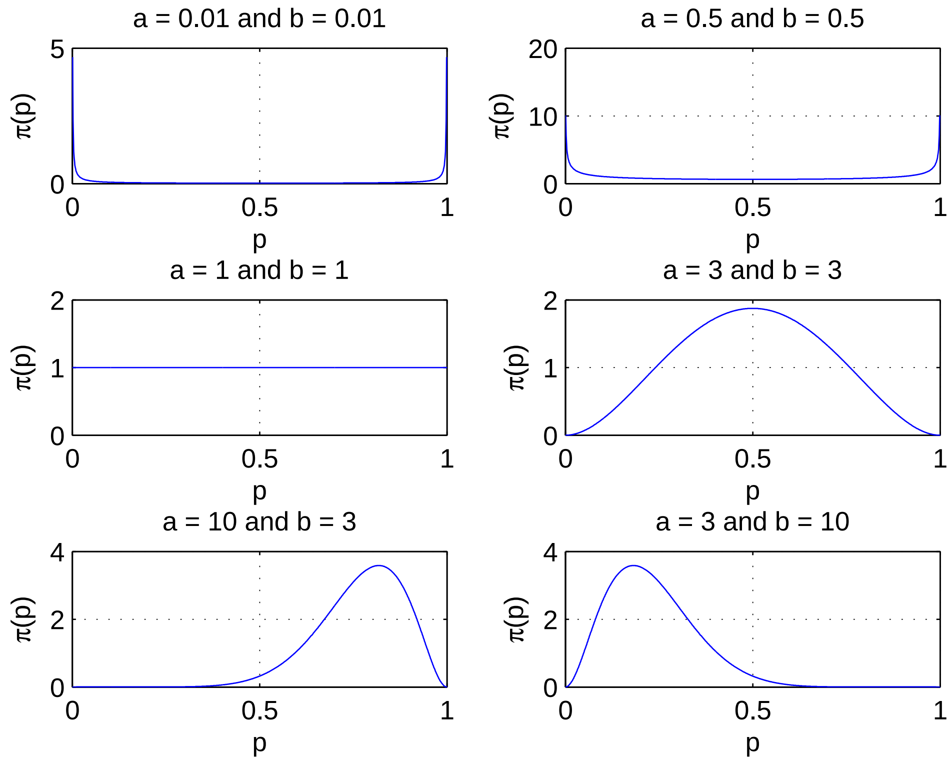

Figure 1: The beta prior for different values of a and b.

Combining Equations 1 and 2 it follows that the posterior distribution of p is a $Beta\left(\sum_{i=1}^{m} x_i + a, mn - \sum_{i=1}^{m} x_i + b\right)$ distribution, i.e.

$$\pi(p|data) = \frac{1}{B\left(\sum_{i=1}^{m} x_i + a, mn - \sum_{i=1}^{m} x_i + b\right)} p^{\sum_{i=1}^{m} x_i + a - 1} (1-p)^{mn - \sum_{i=1}^{m} x_i + b - 1}. \tag{3}$$

If the process remains stable, the control chart limits for a future sample of n Bernoulli trials which results in T successes can be derived. Given n and p, the distribution of T is binomial, and the unconditional prediction distribution of T is

$$\begin{aligned} f(T|data) &= \int_0^1 f(T|p)\,\pi(p|data)\,dp \\ &= \frac{\Gamma(n+1)}{\Gamma(T+1)\Gamma(n-T+1)} \frac{B\left(\sum_{i=1}^{m} x_i + a + T, mn - \sum_{i=1}^{m} x_i + b + n - T\right)}{B\left(\sum_{i=1}^{m} x_i + a, mn - \sum_{i=1}^{m} x_i + b\right)} \quad 0 \le T \le n \end{aligned} \tag{4}$$

which is a beta-binomial distribution. It is assumed that the sample size is the same for the posterior distribution and the future sample. The predictive distribution in Equation 4 can be used to obtain the control chart limits, where the rejection region is defined as $\alpha = \sum_{R^{\star}(\alpha)} f(T\,|data)$. An exact acceptance region of size $1-\alpha$ can be found based on the beta-binomial distribution, (Menzefricke (2002)).

Our prior is just adding $a-1$ successes and $b-1$ failures to the data set. If we have no strong prior beliefs, we can choose a prior that gives equal weight to all possibles values of the unknown parameter. This is also known as a flat or noninformative prior. If we let $a=1$ and $b=1$, we will get such a prior, which is just the uniform distribution over the $[0,1]$ interval. If we let $a={}^{1}\!/_{2}$ and $b={}^{1}\!/_{2}$ we will get the well-known Jeffreys prior, from Jeffreys (1939).

3 Simulation Study

In this simulation study the average run lengths and false alarm rates will be compared using the classical (frequentist) method and the proposed Bayesian method. The predictive density given in Equation 4 will be used to obtain the control chart limits when the Bayesian approach is used. The run length of a control procedure is the number of samples required before an out-of-control signal is given. A good control procedure has a suitably large average run length when the process is in-control and a small average run length otherwise, from Woodall (1985).

The average run length (ARL) is calculated as:

$$\text{ARL} \;=\; \frac{1}{P\left(\text{sample point plots out of control}\right)}.$$

If the process is in-control, the probability that a point plots out-of-control is also known as the false alarm rate (FAR). Montgomery (1996) defines the average run length as the average number of points that must be plotted before a point indicates an out-of-control condition. If the process is in-control, the expected nominal value for the false alarm rate is 0.0027 and the expected nominal value for the average run length is $(0.0027)^{-1} = 370.3704$. That means even if the process remains in control, an out-of-control signal will be generated every 370 samples, on average.

We will consider a number of different samples sizes, n, and number of samples, m. Chakraborti &

Human (2006) considered most of the combinations used in this simulation study, and they determined the average run lengths and false alarm rates using the classical (frequentist) method. The number of simulations is equal to 100 000. The simulations were run in MATLAB®. The results for the given m and n values, using the classical and Bayesian methods, are given in Table 1. For the frequentist method it is assumed that $p = 0.5$ when determining the average run length and the false alarm rate, Chakraborti & Human (2006) also used this value. When using the Bayesian method, the value of p is of course unknown and the prior distribution given in Equation 1 is used. As mentioned, six different priors were considered.

For the simulation procedure, we randomly generated m binomial random variables. The 3-sigma control chart limits were calculated for the classical method. Followed by calculating the false alarm rate and the run length, using the binomial distribution. For the Bayesian method, the control chart limits were calculated using the predictive density given in Equation 4, with rejection region of size $\alpha = 0.0027$. Followed by calculating the the false alarm rate and the run length, using the beta-binomial distribution. This was repeated 100 000 times, and then the average of the false alarm rates was calculated, and also the average of the run lengths.

From Table 1 we can see that in every single case, the unconditional false alarm rate is lower for one of the Bayesian methods. For five of the nineteen cases that we considered, the average run length is bigger for the classical method than for any of the Bayesian methods. Typically, one wants a smaller false alarm rate and a larger average run length. For the Bayesian method, the false alarm rate is generally closer to the nominal level of 0.0027. For example, when $m = 2$ and $n = 250$, the false alarm rate is equal to 0.00274 when a $Beta\,(10,3)$ prior is used. Where the other priors yield a false alarm rate very close to 0.0027, for this case the classical (frequentist) method yields a false alarm rate of 0.01424.

Table 1: (a) Average run lengths and (b) average false alarm rates for different values of m and n, where $p = 0.5$ is used for the Frequentist method.

		Freq	**Bayes** $B(0.01, 0.01)$	**Bayes** $B(0.5, 0.5)$	**Bayes** $B(1, 1)$	**Bayes** $B(3, 3)$	**Bayes** $B(10, 3)$	**Bayes** $B(3, 10)$
$m = 1$ &	(a)	140.0652	349.3294	359.1191	375.5631	341.1348	356.0179	341.9281
$n = 50$	(b)	0.03689	0.00298	0.00290	0.00277	0.00303	0.00287	0.00300
$m = 2$ &	(a)	188.8749	336.4882	338.0909	289.3581	309.4339	327.6048	310.7671
$n = 25$	(b)	0.01832	0.00324	0.00326	0.00364	0.00339	0.00335	0.00357
$m = 5$ &	(a)	462.7528	278.1979	279.8538	281.5882	293.9073	272.4799	205.0873
$n = 10$	(b)	0.00630	0.00590	0.00582	0.00574	0.00538	0.00529	0.00776
$m = 2$ &	(a)	237.7438	316.3985	323.6099	334.5398	323.8041	337.9737	311.0233
$n = 30$	(b)	0.01566	0.00340	0.00332	0.00322	0.00329	0.00320	0.00352
$m = 5$ &	(a)	437.2479	255.2552	256.8947	258.6095	299.9057	284.3192	270.0392
$n = 12$	(b)	0.00642	0.00546	0.00541	0.00535	0.0048	0.00469	0.00530
$m = 4$ &	(a)	259.1217	325.1103	327.80139	317.3218	288.4798	323.5345	306.4846
$n = 25$	(b)	0.00778	0.00331	0.00328	0.00349	0.00377	0.00331	0.00353
$m = 5$ &	(a)	349.0077	278.5296	280.1726	281.8248	285.8987	305.9498	292.8715
$n = 20$	(b)	0.00576	0.00402	0.00399	0.00397	0.00392	0.00365	0.00387
$m = 6$ &	(a)	392.2616	289.5493	290.7965	292.6434	297.5224	312.6546	305.1708
$n = 20$	(b)	0.00514	0.00395	0.00394	0.00391	0.00385	0.00364	0.00378
$m = 4$ &	(a)	252.5432	323.83550	316.6086	318.8539	326.0209	316.1081	310.0659
$n = 30$	(b)	0.00763	0.00337	0.00346	0.00343	0.00335	0.00347	0.00354
$m = 4$ &	(a)	259.3864	339.6836	341.1857	352.6781	339.7289	347.0327	345.0694
$n = 50$	(b)	0.00708	0.00304	0.00303	0.00295	0.00303	0.00295	0.00297
$m = 10$ &	(a)	408.9574	302.9630	303.4763	303.9957	305.9098	321.7337	313.7103
$n = 20$	(b)	0.00374	0.00389	0.00388	0.00388	0.00385	0.00365	0.00378
$m = 2$ &	(a)	187.8679	353.0119	350.8284	356.2148	363.9197	355.2329	354.4174
$n = 125$	(b)	0.01435	0.00287	0.00289	0.00285	0.00279	0.00284	0.00285
$m = 5$ &	(a)	285.1554	339.3359	340.3523	341.3812	340.7811	339.8927	334.7293
$n = 50$	(b)	0.00614	0.00307	0.00306	0.00305	0.00306	0.00309	0.00314
$m = 3$ &	(a)	201.6699	356.8151	358.2277	359.4852	359.2769	348.7078	346.7044
$n = 100$	(b)	0.01003	0.00285	0.00283	0.00282	0.00282	0.00288	0.00290
$m = 6$ &	(a)	276.9894	346.9128	344.0441	344.0641	347.0209	343.9648	340.6412
$n = 50$	(b)	0.00533	0.00304	0.00306	0.00306	0.00307	0.00306	0.00309
$m = 2$ &	(a)	187.3209	354.8408	358.9326	359.4457	363.4456	367.0650	367.0676
$n = 250$	(b)	0.01424	0.00284	0.00281	0.00280	0.00277	0.00274	0.00274
$m = 10$ &	(a)	328.6722	333.2081	333.4760	331.1047	332.1746	339.1109	337.6287
$n = 50$	(b)	0.00399	0.00310	0.00309	0.00313	0.00312	0.00304	0.00306
$m = 5$ &	(a)	239.1253	356.4929	356.8573	357.2285	356.7893	357.5209	356.9173
$n = 150$	(b)	0.00630	0.00282	0.00281	0.00281	0.00282	0.00281	0.00282
$m = 10$ &	(a)	291.1962	342.0943	342.2831	342.4749	342.4087	344.8229	344.4709
$n = 75$	(b)	0.00417	0.00297	0.00296	0.00296	0.00296	0.00294	0.00295

4 Illustrative Example

Consider the following example from Montgomery (1996), Example 6-1 on page 255. Chakraborti & Human (2006) also considered this example. Frozen orange juice concentrate is packed in 6-oz cardboard cans. These cans are formed on a machine by spinning them from cardboard stock and attaching a metal bottom panel. By inspection of a can, we may determine whether, when filled, it could possibly leak either on the side seam or around the bottom joint. Such a nonconforming can has an improper seal on either the side seam or the bottom panel. We wish to set up a control chart to improve the proportion of nonconforming cans produced by this machine. To establish the control chart, 30 samples of $n = 50$ cans each were selected at half-hour intervals over a three-shift period in which the machine was in continuous operation. Once the control chart was established, samples 15 and 23 were found to be out-of-control, and eliminated after further investigation. Revised limits were calculated using the remaining samples, with $m = 28$ and $n = 50$. Based on the revised control limits, sample 21 was found to be out-of-control. Since further investigations regarding sample 21 did not produce any reasonable or logical assignable cause, it was not discarded.

For this given data set $\sum_{i=1}^{28} x_i = 301$, the total number of nonconforming cans after discarding samples 15 and 23, is observed. Considering this example, we have, $n = 50$, $m = 28$ and $\sum_{i=1}^{m} x_i = 301$. If we use a beta prior, $p \sim Beta(a,b)$, the posterior distribution for this example will be a $Beta(301+a, 1\,099+b)$. Chakraborti & Human (2006) determined the frequentist control limits assuming $p = 0.2$, when determining the average run length and the false alarm rate. From the data given in this example, $\overline{p} = 0.215$. We also calculated the frequentist control limits assuming different values for p, close to $\overline{p}$. For the frequentist method we considered the cases where it is assumed that $p = 0.18$, 0.2, 0.22 and 0.24, when determining the average run length and the false alarm rate. For the Bayesian method we considered the six different priors, and used the predictive density from Equation 4 to obtain the control chart limits, false alarm rate and average run length. Figure 2 shows different plots of the posterior distribution for the six different priors considered. Figure 3 shows bar graphs of the predictive density function, $f(T\,|data)$, where T is the number of nonconformities in a future sample. The results are given in Table 2.

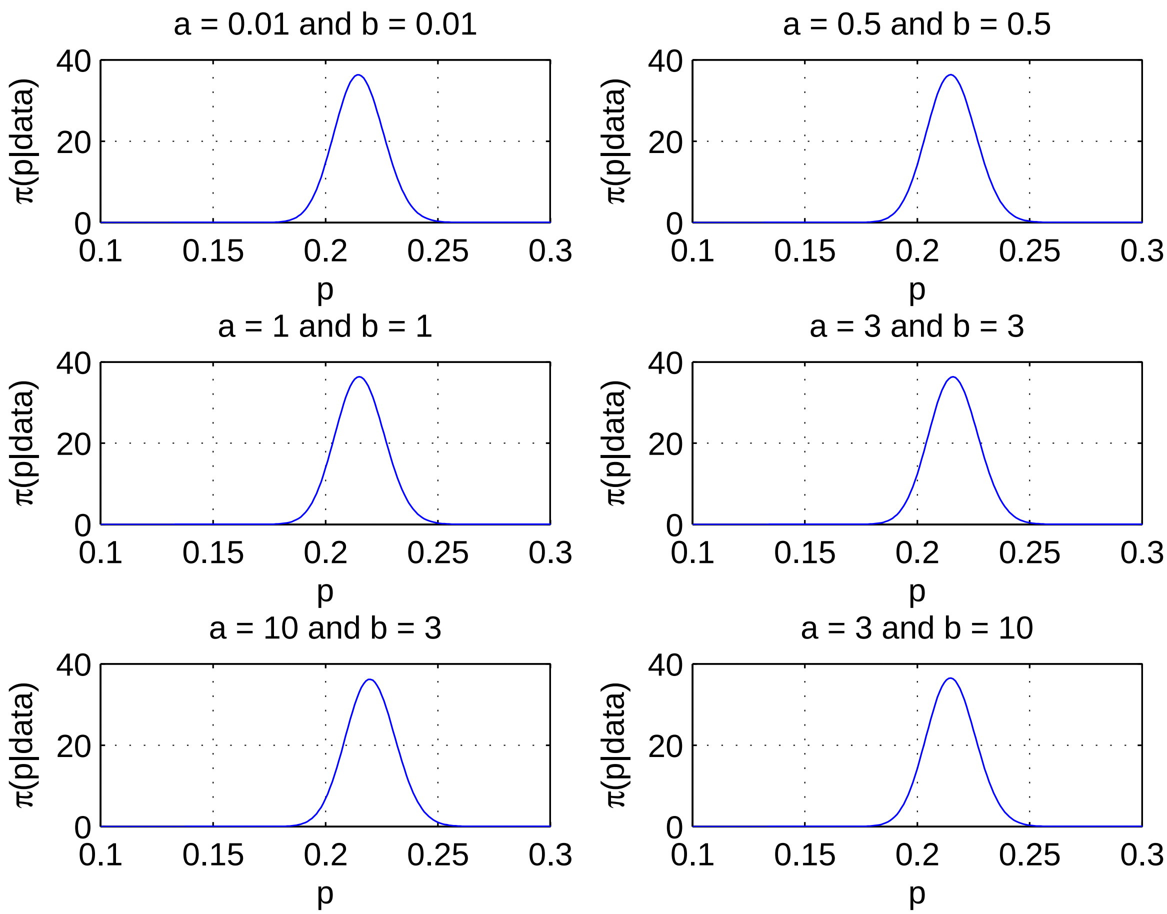

Figure 2: Posterior distribution of p, when $n = 50$, $m = 28$ and $\sum_{i=1}^{m} x_i = 301$ for different values of a and b.

Table 2: Lower control limits, upper control limits, average run lengths and false alarm rates given $n = 50$, $m = 28$ and $\sum_{i=1}^{m} x_i = 301$.

	$\langle nLCL \rangle$	$\langle nUCL \rangle$	ARL	FAR
Freq - $p = 0.18$	2	19	269.7451	0.0037
Freq - $p = 0.2$	2	19	450.8868	0.0022
Freq - $p = 0.22$	2	19	278.8566	0.0036
Freq - $p = 0.24$	2	19	111.6601	0.0090
Bayes - $B(0.01, 0.01)$	3	20	233.7005	0.0043
Bayes - $B(0.5, 0.5)$	3	20	234.5995	0.0043
Bayes - $B(1, 1)$	3	20	235.5000	0.0042
Bayes - $B(3, 3)$	3	20	238.9261	0.0042
Bayes - $B(10, 3)$	3	21	326.4087	0.0031
Bayes - $B(3, 10)$	2	20	234.6244	0.0043

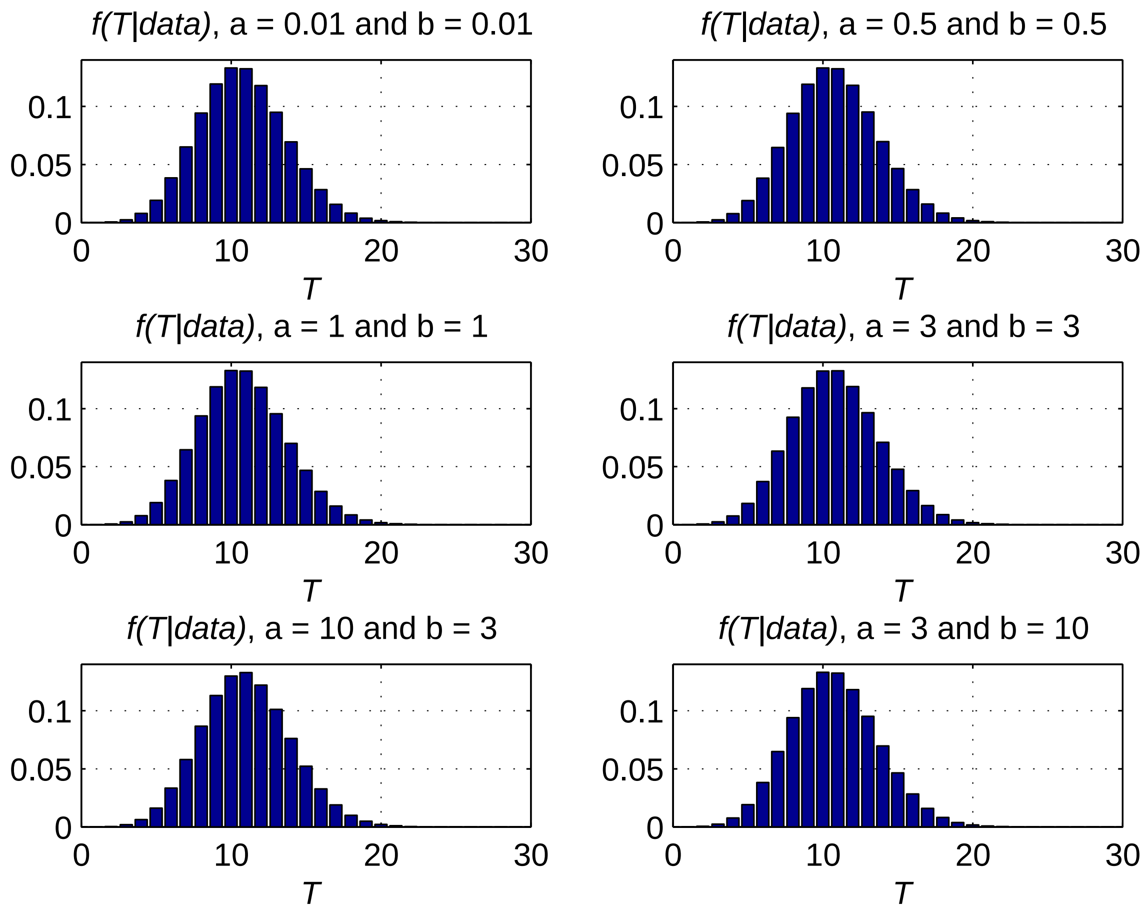

Figure 3: Bar graph of the predictive density of $f(T\,|data)$ for different values of a and b.

In Table 2, $\langle nLCL \rangle$ denotes the largest integer not exceeding $nLCL$ and $\langle nUCL \rangle$ denotes the largest integer not exceeding $nUCL$. The results are given in Table 2. When we consider the classical method assuming $p = 0.2$, the average run length is the largest and the false alarm rate the smallest. Notice how drastically these values change when we assume a different value for p. Using the Bayesian method with a $Beta(10,3)$ prior we obtain an average run length equal to 326.4087, which is the closest to the expected nominal value of 370. Using this prior we obtain a false alarm rate equal to 0.0031, which is the closest to the expected nominal value of 0.0027. When this prior is used, the Bayesian method gives wider control limits than those obtained from the classical method.

5 Conclusion

The usual operation of the p - chart was extended by introducing a Bayesian approach for the p - chart. A conjugate prior, the beta distribution, was used. Where we considered six different priors. We conclude that the proposed Bayesian method gives wider control limits than those obtained from the classical method. The Bayesian method generally gives larger values for the average run length and smaller values for the false alarm rate. A smaller value for the false alarm rate is desired. As stated in Woodward & Naylor (1993), the use of Bayesian methods has the advantage that the knowledge of the process gained from experience can be incorporated into the methodology through the prior distributions for the process settings and adjustments.

References

Calabrese, J. M. 1995. Bayesian process control for attributes. *Management science*, **41**(4), 637 – 645.

Chakraborti, S., & Human, S. W. 2006. Parameter estimation and performance of the p - chart for attributes data. *Ieee transactions on reliability*, **55**(3), 559 – 566.

Hamada, M. 2002. Bayesian tolerance interval control limits for attributes. *Quality and reliability engineering international*, **18**(1), 45 – 52.

Jeffreys, H. 1939. *Theory of probability*. 1st edn. Oxford University Press, Oxford.

Jensen, W. A., Jones-Farmer, L. A., Champ, C. W., & Woodall, W. H. 2006. Effects of parameter estimation on control chart properties: A literature review. *Journal of quality technology*, **38**(4), 349 – 364.

Menzefricke, U. 2002. On the evaluation of control chart limits based on predictive distributions. *Communications in statistics - theory and methods*, **31**(8), 1423 – 1440.

Montgomery, D. C. 1996. *Introduction to statistical quality control*. 3rd edn. John Wiley & Sons, New York.

Woodall, W. H. 1985. The statistical design of quality control charts. *The statistician*, **34**, 155 – 160.

Woodward, P. W., & Naylor, J. C. 1993. An application of Bayesian methods in SPC. *The statistician*, **42**, 461 – 469.