A Bayesian Control Chart for a One-sided Upper Tolerance Limit for the Two-parameter Exponential Distribution

R. van Zyl[1], A.J. van der Merwe[2]

[1]Quintiles International, ruaanvz@gmail.com

[2]University of the Free State

Abstract

By using failure mileages of military carriages by Grubbs (1971) and Krishnamoorthy and Mathew (2009) a Bayesian procedure is applied to obtain control limits for a one-sided upper tolerance limit for the the two-parameter exponential distribution. The Jeffreys' prior $p(\theta, \mu) \propto \theta^{-1}$ is used and it is shown that the posterior distributions of μ and θ match the generalized pivotal quantities for μ and θ. By using simulation methods for the one-sided tolerance limit, "run-lengths" and average "run-lengths" are derived. This article illustrates the flexibility and unique features of Bayesian simulation for obtaining the posterior predictive distribution of a future one-sided tolerance limit.

Keywords: Jeffreys' prior, two-parameter exponential, tolerance limits, run-length, control chart

1 Introduction

The two-parameter exponential distribution plays an important role in engineering, life testing and medical sciences. In these studies where the data are positively skewed, the exponential distribution is as important as the normal distribution is in sampling theory and agricultural statistics. Researchers have studied various aspects of estimation and inference for the two-parameter exponential distribution using either the frequentist approach or the Bayesian procedure.

However, while parameter estimation and hypothesis testing related to the two-parameter exponential distribution are well documented in the literature, the research on control charts has received little attention. Ramalhoto and Morais (1999) developed a control chart for monitoring the scale parameter while Sürücü and Sazak (2009) presented a control chart scheme in which moments are used. Mukherjee, McCracken, and Chakraborti (2014) on the other hand proposed several control charts and monitoring schemes for the location and the scale parameters of the two-parameter exponential distribution.

In this paper a one-sided upper tolerance limit for the two-parameter exponential distribution will be developed by using generalized pivotal quantities and a Bayesian procedure.

Bayarri and García-Donato (2005) give the following reasons for recommending a Bayesian analysis:

- Control charts are based on future observations and Bayesian methods are very natural for prediction.
- Uncertainty in the estimation of the unknown parameters is adequately handled.
- Implementation with complicated models and in a sequential scenario poses no methodological difficulty, the numerical difficulties are easily handled via Monte Carlo methods;
- Objective Bayesian analysis is possible without introduction of external information other than the model, but any kind of prior information can be incorporated into the analysis, if desired.

Krishnamoorthy and Mathew (2009) and Hahn and Meeker (1991) defined a tolerance interval as an interval that is constructed in such a way that it will contain a specified proportion or more of the population with a certain degree of confidence. The proportion is also called the content of the tolerance interval. As opposed to confidence intervals that give information on unknown population parameters, a one-sided upper tolerance limit for example provides information about a quantile of the population.

2 Preliminary and Statistical Results

In this section the same notation will be used as given in Krishnamoorthy and Mathew (2009).

The two-parameter exponential distribution has the probability density function

$$f(x;\mu,\theta) = \frac{1}{\theta}\exp\left\{-\frac{(x-\mu)}{\theta}\right\} \ x > \mu, \ -\infty < \mu < \infty, \ \theta > 0$$

where μ is the location parameter and θ is the scale parameter. Krishnamoorthy and Mathew (2009) defined the two-parameter exponential for $\mu > 0$. Our definition is for $-\infty < \mu < \infty$ and therefore differs somewhat from theirs. In the literature, see for example Johnson and Kotz (1970), where the two-parameter exponential has been defined for $-\infty < \mu < \infty$.

Let $X_1, X_2, \ldots, X_n$ be a sample of n observations from the two-parameter exponential distribution. The maximum likelihood estimators for μ and θ are given by

$$\hat{\mu} = X_{(1)}$$

and

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - X_{(1)}\right) = \bar{X} - X_{(1)}$$

where $X_{(1)}$ is the minimum or the first order statistic of the sample. It is well known (see Johnson and Kotz (1970); Lawless (1982); Krishnamoorthy and Mathew (2009)) that $\hat{\mu}$ and $\hat{\theta}$ are independently distributed with

$$\frac{(\hat{\mu}-\mu)}{\theta} \sim \frac{\chi_2^2}{2n} \text{ and } \frac{\hat{\theta}}{\theta} \sim \frac{\chi_{2n-2}^2}{2n} \tag{2.1}$$

Let $\hat{\mu}_0$ and $\hat{\theta}_0$ be observed values of $\hat{\mu}$ and $\hat{\theta}$ then it follows from Equation 2.1 that a generalized pivotal quantity (GPQ) for μ is given by

$$G_\mu = \hat{\mu}_0 - \frac{\chi_2^2}{\chi_{2n-2}^2}\hat{\theta}_0 \tag{2.2}$$

and a GPQ for θ is given by

$$G_\theta = \frac{2n\hat{\theta}_0}{\chi_{2n-2}^2} \tag{2.3}$$

From a Bayesian perspective it will be shown that G_μ and G_θ are actually the posterior distributions of μ and θ if the prior $p(\mu,\theta) \propto \theta^{-1}$ is used.

3 Tolerance Limits - Generalized Variable Approach

Theorem 3.1. *The p quantile of a two-parameter exponential distribution is given by $q_p = \mu - \theta \ln(1-p)$.*

Proof. The proof is provided in Appendix A. ◻

By replacing the parameters by their GPQ's a GPQ for q_p can be obtained and is given by

$$
\begin{aligned}
G_{q_p} &= G_p - G_\theta \ln(1-p) \\
&= \hat{\mu}_0 - \left[\frac{\chi_2^2 + 2n\ln(1-p)}{\chi_{2n-2}^2}\right] \hat{\theta}_0 .
\end{aligned}
$$

Let $E_{p;\alpha}$ denotes the α quantile of $E_p = \frac{\chi_2^2 + 2n\ln(1-p)}{\chi_{2n-2}^2}$, then as mentioned by Krishnamoorthy and Mathew (2009)

$$\hat{\mu} - E_{p;\alpha}\hat{\theta}_0$$

is a $1-\alpha$ upper confidence limit for q_p, which means that $(p, 1-\alpha)$ is an upper tolerance limit for the *exponential* (μ, θ) distribution. Also

$$\hat{\mu}_0 - E_{1-p,1-\alpha}\hat{\theta}_0$$

is a $1-\alpha$ tolerance limit for $q_{1-p} = \mu - \theta \ln p$, or equivalently a $(p, 1-\alpha)$ lower tolerance limit for the *exponential* (μ, θ) distribution.

It is shown in Roy and Mathew (2005) and Krishnamoorthy and Mathew (2009) that the upper and lower tolerance limits obtained using the generalized variable approach are actually exact, which means that they have the correct frequentist coverage probabilities.

4 Bayesian Procedure

In this section it will be shown that the Bayesian procedure is the same as the generalized variable approach.

If a sample of n observations are drawn from the two-parameter exponential distribution, then the likelihood function is given by

$$L(\mu, \theta | data) = \left(\frac{1}{\theta}\right)^n \exp\left\{-\frac{1}{\theta}\sum_{i=1}^{n}(x_i - \mu)\right\}.$$

As prior the Jeffreys' prior

$$p(\mu, \theta) \propto \theta^{-1}$$

will be used.

The joint posterior distribution of μ and θ is

$$
\begin{aligned}
p(\theta, \mu | data) &\propto p(\mu, \theta) L(\mu, \theta | data) \\
&= K_1 \left(\tfrac{1}{\theta}\right)^{n+1} \exp\left\{-\tfrac{n}{\theta}(\bar{x} - \mu)\right\} \quad -\infty < \mu < x_{(1)},\ 0 < \theta < \infty
\end{aligned}
\tag{4.1}
$$

It can easily be shown that

$$K_1 = \frac{n^n \left(\hat{\theta}\right)^{n-1}}{\Gamma\left(n-1\right)} \text{ where } \hat{\theta} = \bar{x} - x_{(1)}.$$

The posterior distribution of μ is

$$\begin{aligned} p\left(\mu|data\right) &= \int_0^\infty p\left(\theta,\mu|data\right)d\theta \\ &= \left(n-1\right)\left(\hat{\theta}\right)^{n-1}\left(\tfrac{1}{\bar{x}-\mu}\right)^n \end{aligned} \tag{4.2}$$

and the posterior distribution of θ is

$$\begin{aligned} p\left(\theta|data\right) &= \int_{-\infty}^{x_{(1)}} p\left(\theta,\mu|data\right)d\mu \\ &= K_2\left(\tfrac{1}{\theta}\right)^n \exp\left\{-\tfrac{n\hat{\theta}}{\theta}\right\} \text{ where } 0<\theta<\infty \end{aligned} \tag{4.3}$$

an Inverse Gamma distribution where

$$K_2 = \frac{n^{n-1}\left(\hat{\theta}\right)^{n-1}}{\Gamma\left(n-1\right)}.$$

The conditional posterior distribution of θ given μ is given by

$$\begin{aligned} p\left(\theta|\mu,data\right) &= \tfrac{p(\theta,\mu|data)}{p(\mu|data)} \\ &= K_3\left(\tfrac{1}{\theta}\right)^{n+1}\exp\left\{-\tfrac{n}{\theta}\left(\bar{x}-\mu\right)\right\} \text{ where } 0<\theta<\infty \end{aligned} \tag{4.4}$$

an Inverse Gamma distribution where

$$K_3 = \frac{\left\{n\left(\bar{x}-\mu\right)\right\}^n}{\Gamma\left(n\right)}.$$

Also the conditional posterior distribution of μ given θ is

$$\begin{aligned} p\left(\mu|\theta,data\right) &= \tfrac{p(\theta,\mu|data)}{p(\theta|data)} \\ &= \tfrac{n}{\theta}\exp\left\{-\tfrac{n}{\theta}\left(x_{(1)}-\mu\right)\right\} \text{ where } -\infty<\mu<x_{(1)} \end{aligned} \tag{4.5}$$

The following theorem can now easily be proved:

Theorem 4.1. *The distribution of the generalized pivotal quantities G_μ and G_θ defined in Equations 2.2 and 2.3 are exactly the same as the posterior distributions $p\left(\mu|data\right)$ and $p\left(\theta|data\right)$ given in Equations 4.2 and 4.3.*

Proof. The proof is given in Appendix B. □

5 The Predictive Distribution of a Future Sample One-sided Upper Tolerance Limit

Consider a future sample of m observations from the two-parameter exponential population: $X_{1f}, X_{2f}, \ldots, X_{mf}$. The future sample mean is defined as $\bar{X}_f = \frac{1}{m}\sum_{j=1}^{m} X_{jf}$. The smallest value in the sample is denoted by $\hat{\mu}_f$ and $\hat{\theta}_f = \bar{X}_f - \hat{\mu}_f$. For given θ and μ the following distributions follow (see Equation 2.1):

$$\hat{\mu}_f \sim \theta \frac{\tilde{\chi}_2^2}{2m} + \mu \tag{5.1}$$

and

$$\hat{\theta}_f \sim \frac{\tilde{\chi}_{2m-2}^2}{2m}\theta \tag{5.2}$$

where $\tilde{\chi}_2^2$ and $\tilde{\chi}_{2m-2}^2$ denote chi-square random variables with 2 and $2m-2$ degrees of freedom.

Therefore

$$U_f = \hat{\mu}_f - \tilde{k}_2 \hat{\theta}_f$$

is an upper future confidence limit (tolerance limit) for

$$q_p = \mu - \theta \ln(1-p)$$

the p quantile of the two-parameter exponential distribution where $\tilde{k}_2 = E_{p,\alpha}^f$ denotes the α quantile of

$$E_p^f = \frac{\tilde{\chi}_2^2 + 2m\ln(1-p)}{\tilde{\chi}_{2m-2}^2}.$$

The predictive distribution of U_f can easily be obtained by simulation. From Equations 5.1 and 5.2 it follows that

$$\begin{aligned} U_f|\mu,\theta &\sim \theta\tfrac{\tilde{\chi}_2^2}{2m} + \mu - \tilde{k}_2\tfrac{\tilde{\chi}_{2m-2}^2}{2m}\theta \\ &\sim \mu + \tfrac{\theta}{2,}\left(\tilde{\chi}_2^2 - \tilde{k}_2\tilde{\chi}_{2m-2}^2\right) \end{aligned} \tag{5.3}$$

We are however interested in the unconditional predictive distribution of an upper future tolerance limit, i.e., of $U_f|data$. By using either the generalized pivotal quantity approach or the Bayesian procedure it follows using Equations 2.2 and 2.3 that

$$U_f|data \sim \hat{\mu}_0 - \frac{\hat{\theta}_0}{\chi_{2n-2}^2}\left\{\chi_2^2 - \frac{n}{m}\left(\tilde{\chi}_2^2 - \tilde{k}_2\tilde{\chi}_{2m-2}^2\right)\right\} \tag{5.4}$$

From Equation 5.4 it can be seen that the exact distribution of U_f will be quite complicated. An approximation of the distribution can however be obtained by using the following Monte Carlo simulation procedure:

1. Simulate χ_2^2, $\tilde{\chi}_2^2$, χ_{2n-2}^2 and $\tilde{\chi}_{2m-2}^2$ and substitute the simulated values in Equation 5.4.
2. Repeat (1) a large number of times, say $l = 1000000$ times.

Although the exact distribution of U_f is difficult to obtain analytically, the exact moments of U_f can be derived.

The following theorem can now be proved:

Theorem 5.1. *The exact mean and variance of U_f is given by*

$$E\left(U_f|data\right) = \hat{\mu}_0 - \frac{\hat{\theta}_0}{(n-2)}\left\{1 - \frac{n}{m}\left[1 - \tilde{k}_2\left(m-1\right)\right]\right\} \tag{5.5}$$

and

$$Var\left(U_f|data\right) = \frac{\hat{\theta}_0^2}{(n-2)(n-3)}\left\{1 + \left(\frac{n}{m}\right)^2\left[1 + \tilde{k}_2^2\left(m-1\right)\right] + \frac{1}{(n-2)}\left\{1 - \frac{n}{m}\left[1 - \tilde{k}_2\left(m-1\right)\right]\right\}^2\right\} \tag{5.6}$$

Proof. The proof is given in Appendix C. □

6 Example

The following data is given in Grubbs (1971) as well as in Krishnamoorthy and Mathew (2009). The failure mileages given in Table 6.1 fit a two-parameter exponential distribution.

Table 6.1: Failure Mileages of 19 Military Carriers

162	200	271	302	393	508	539	629	706	777
884	1008	1101	1182	1463	1603	1984	2355	2880	

For this data, the estimates are $\hat{\mu} = x_{(1)} = 162$, $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - x_{(1)}\right) = \bar{x} - x_{(1)} = 835.21$ and $n = 19$.

As mentioned in the introductory section the aim of this article is to obtain a control chart for a one-sided upper tolerance limit in the case of the two-parameter exponential distribution. It was also mentioned that control charts are based on future observations and Bayesian methods are very natural.

For $m = 2$, $k_2 = -70.6745$ and by using $10,000,000$ Monte Carlo simulations, the posterior predictive distribution of U_f defined in Equation 5.4 is given in Figure 1. Figure 1 is therefore the distribution of an upper future tolerance limit for the mileages of the next two military personal carriers that will fail in service.

[Figure 1 about here.]

In Table 6.2 it is shown that the calculated means and variances using the simulation method or the formulae do not differ much.

Table 6.2: Mean and Variance of U_f

	$E\left(U_f\|data\right)$	$Var\left(U_f\|data\right)$
Simulation	33554	1.2246×10^9
Formulae	33565.867	1.2264×10^9

7 Control Chart for a Future One-sided Upper Tolerance Limit

It is well known that statistical quality control is actually implemented in two phases. In Phase I the primary interest is to assess process stability. The practitioner must therefore be sure that the process is in statistical control before control limits can be determined for online monitoring of the process in Phase II.

By using the predictive distribution of U_f a Bayesian procedure will be developed to obtain a control chart for a future one-sided upper tolerance limit. Assuming that the process remains stable, the predictive distribution can be used the derive the distribution of the "run-length" and average "run-length". From Figure 1 it follows that a 99.73% upper control limit for $U_f = \hat{\mu}_f - \tilde{k}_2 \hat{\theta}_f$ is 218550. Therefore the rejection region of size $\beta\,(\beta = 0.0027)$ for the predictive distribution is

$$\beta = \int_{R(\beta)} f\left(U_f | data\right) dU_f$$

where $R(\beta)$ represents those values of U_f that are larger than 218550.

The "run-length" is defined as the number of future U_f values (r) until the control chart signals for the first time (Note that r does not include that U_f value when the control chart signals). Given μ and θ and a stable Phase I process, the distribution of the "run-length" r is geometric with parameter

$$\psi(\mu, \theta) = \int_{R(\beta)} f\left(U_f | \mu, \theta\right) dU_f$$

where $f\left(U_f | \mu, \theta\right)$ is the distribution of U_f given that μ and θ are known. The values of μ and θ are however unknown and the uncertainty of these parameter values are described by their joint posterior distribution $p(\theta, \mu | data)$ given in Equation 4.1.

By simulating μ and θ from $p(\theta, \mu | data)$ the probability density function of $f\left(U_f | \mu, \theta\right)$ as well as the parameter $\psi(\mu, \theta)$ can be obtained. This must be done for each future sample. In other words, for each future sample μ and θ must first be simulated from $p(\theta, \mu | data)$ and then $\psi(\mu, \theta)$ calculated. Therefore, by simulating all possible combinations of μ and θ from their joint posterior distribution a large number of $\psi(\mu, \theta)$ values can be obtained. Also, a large number of geometric distributions, i.e., a large number of "run-length" distributions each with a different parameter value $(\psi(\mu_1, \theta_1), \psi(\mu_2, \theta_2), \ldots, \psi(\mu_m, \theta_m))$ can be obtained.

As mentioned the "run-length" r for given μ and θ is geometrically distributed with mean

$$E(r | \mu, \theta) = \frac{1 - \psi(\mu, \theta)}{\psi(\mu, \theta)}$$

and variance

$$Var(r | \mu, \theta) = \frac{1 - \psi(\mu, \theta)}{\psi^2(\mu, \theta)}.$$

The unconditional moments, $E(r | data)$, $E\left(r^2 | data\right)$ and $Var(r | data)$ can therefore be obtained by simulation or numerical integration. For further details refer to Menzefricke (2002, 2007, 2010a,b).

In Figure 2 the predictive distribution of the "run-length" is displayed for the 99.73% upper control limit. As mentioned, for given μ and θ the "run-length" r is geometric with parameter $\psi(\mu, \theta)$. The unconditional "run-length" as displayed in Figure 2 is therefore obtained using the Rao-Blackwell method, i.e., the average of a large number of conditional "run-lengths".

[Figure 2 about here.]

Define $\bar{\psi}(\mu, \theta) = \frac{1}{m}\sum_{i=1}^{m}\psi(\mu_i, \theta_i)$. From Menzefricke (2002) it follows that if $m \to \infty$; then $\bar{\psi}(\mu, \theta) \to 0.0027$ and the harmonic mean of $r \to (0.0027)^{-1} = 370$. From Figure 2 it can be seen that the harmonic mean of r is 366.65, which is close to 370.

In Figure 3 the distribution of the average "run-length" is given.

[Figure 3 about here.]

For known μ and θ the expected "run-length" is $\frac{1}{0.0027} = 370$. If μ and θ are unknown and estimated from the posterior distribution the expected "run-length" will usually be larger than 370, especially if the sample size is small.

8 Conclusion

This paper develops a Bayesian control chart for monitoring an upper one-sided tolerance limit from the two-parameter exponential distribution. In the Bayesian approach prior knowledge about the unknown parameters is formally incorporated into the process of inference by assigning a prior distribution to the parameters. The information contained in the prior is combined with the likelihood function to obtain the posterior distribution. By using the posterior distribution the predictive distribution of an upper one-sided tolerance limit for the two-parameter exponential can be obtained.

This paper has also shown that the use of the Jeffreys' prior, posterior distributions of $p(\mu|data)$ and $p(\theta|data)$ are exactly equal to the generalized pivotal quantities for μ and θ.

The theory and results described in this paper have been applied to the failure mileages for military carriers analyzed by Grubbs (1971) and Krishnamoorthy and Mathew (2009). The example illustrates the flexibility and unique features of the Bayesian simulation method for obtaining posterior distributions and "run-lengths". This article also illustrated that the harmonic mean resulted in a mean "run-length" close to the 370 that is expected from a 99.73% upper control limit.

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Appendices

A Proof of Theorem 3.1

Let
$$I = \int_{\mu}^{q_p} \frac{1}{\theta} \exp\left\{-\frac{(x-\mu)}{\theta}\right\} dx.$$

Substitute $\frac{(x-\mu)}{\theta} = z$. Therefore $x = \theta z + \mu$ and $dx = \theta dz$

If $x = \mu$ it follows that $z = 0$ and if $x = q_p$ it follows that $z = \frac{(q_p-\mu)}{\theta}$.

Therefore
$$I = \int_0^{\frac{(q_p-\mu)}{\theta}} \exp\left(-z\right) dz = p$$

which means that
$$\left[-\exp\left(-z\right)\right]_0^{\frac{(q_p-\mu)}{\theta}} = p$$

and
$$-\exp\left(-\frac{(q_p-\mu)}{\theta}\right) + 1 = p.$$

Therefore
$$q_p = -\theta \ln\left(1-p\right) + \mu$$

B Proof of Theorem 4.1

(a) The posterior distribution $p\left(\theta|data\right)$ is exactly the same as the distribution of the pivotal quantity $G_\theta = \frac{2n\hat{\theta}}{\chi^2_{2n-2}}$.

Proof:

Let $Z \sim \chi^2_{2n-2}$

$\therefore f\left(z\right) = \frac{1}{2^{n-2}\Gamma(n-1)} z^{n-2} \exp\left\{-\frac{1}{2}z\right\}$

We are interested in the distribution of $\theta = \frac{2n\hat{\theta}}{Z}$.

Therefore $Z = \frac{2n\hat{\theta}}{\theta}$ and $\left|\frac{dZ}{d\theta}\right| = \frac{2n\hat{\theta}}{Z}$.

From this it follows that

$$\begin{aligned} f\left(G_\theta\right) &= f\left(\theta\right) = \frac{1}{2^{n-1}\Gamma(n-1)} \left(\frac{2n\hat{\theta}}{\theta}\right)^{n-2} \frac{2n\hat{\theta}}{\theta^2} \exp\left\{-\frac{n\hat{\theta}}{\theta}\right\} \\ &= \frac{n^{n-1}\left(\hat{\theta}\right)^{n-1}}{\Gamma(n-1)} \left(\frac{1}{\theta}\right)^n \exp\left\{-\frac{n\hat{\theta}}{\theta}\right\} = p\left(\theta|data\right) \end{aligned}$$

See Equation 4.3.

(b) The posterior distribution $p(\mu|data)$ is exactly the same as the distribution of the pivotal quantity $G_\mu = \hat{\mu}_0 - \frac{\chi_2^2}{\chi_{2n-2}^2}\hat{\theta}_0$.

Proof:

Let $F = \frac{\chi_2^2/2}{\chi_{2n-2}^2/(2n-2)} \sim F_{2,2n-2}$

$\therefore g(f) = \left(1 + \frac{1}{n-1}f\right)^{-n}$ where $0 < f < \infty$

We are interested in the distribution of $\mu = \hat{\mu}_0 - \frac{2\hat{\theta}_0}{2n-2}F$ which means that $F = \frac{(n-1)}{\hat{\theta}_0}(\hat{\mu}_0 - \mu)$ and $\left|\frac{dF}{d\mu}\right| = \frac{(n-1)}{\hat{\theta}_0}$.

Therefore

$$\begin{aligned} g(\mu) &= \left\{1 + \frac{1}{\hat{\theta}_0}(\hat{\mu}_0 - \mu)\right\}^{-n} \frac{n-1}{\hat{\theta}_0} \\ &= (n-1)\hat{\theta}_0^{n-1}\left(\frac{1}{\bar{x}-\mu}\right)^n \text{ where } -\infty < \mu < \hat{\mu}_0 \\ &= p(\mu|data) \end{aligned}$$

See Equation 4.2.

(c) The posterior distribution of p$(\mu|\theta, data)$ is exactly the same as the distribution of the pivotal quantity $G_{\mu|\theta} = \hat{\mu}_0 - \frac{\chi_2^2}{2n}\theta$ (see Equation 2.1).

Proof:

Let $\tilde{Z} \sim \chi_2^2$ then

$g(\tilde{z}) = \frac{1}{2}\exp\left\{-\frac{1}{2}\tilde{z}\right\}$.

Let $\mu = \hat{\mu} - \frac{\tilde{z}}{2n}\theta$, then $\tilde{z} = \frac{2n}{\theta}(\hat{\mu}_0 - \mu)$ and $\left|\frac{d\tilde{z}}{d\mu}\right| = \frac{2n}{\theta}$.

Therefore

$$\begin{aligned} g(\mu|\theta) &= \frac{n}{\theta}\exp\left\{-\frac{n}{\theta}(\hat{\mu}_0 - \mu)\right\} \qquad -\infty < \mu < \hat{\mu}_0 \\ &= p(\mu|\theta, data) \end{aligned}$$

See Equation 4.5.

C Proof of Theorem 5.1

$$U_f|data \sim \hat{\mu}_0 - \frac{\hat{\theta}_0}{\chi_{2n-2}^2}\left\{\chi_2^2 - \frac{n}{m}\left(\tilde{\chi}_2^2 - \tilde{k}_2\tilde{\chi}_{2m-2}^2\right)\right\}$$

χ_2^2, $\tilde{\chi}_2^2$, χ_{2n-2}^2 and $\tilde{\chi}_{2m-2}^2$ are all independently distributed. If expected values are taken with respect to the chi-square distribution, it follows that

$$E\left(U_f|\chi_{2n-2}^2, data\right) = \hat{\mu}_0 - \frac{\hat{\theta}_0}{\chi_{2n-2}^2}\left\{2 - \frac{n}{m}\left[2 - \tilde{k}_2(2m-2)\right]\right\}$$

and therefore

$$\begin{aligned} E\left(U_f|data\right) &= \hat{\mu}_0 - \tfrac{\hat{\theta}_0}{2n-4}\left\{2 - \tfrac{n}{m}\left[2 - \tilde{k}_2\left(2m-2\right)\right]\right\} \\ &= \hat{\mu}_0 - \tfrac{\hat{\theta}_0}{(n-2)}\left\{1 - \tfrac{n}{m}\left[1 - \tilde{k}_2\left(m-2\right)\right]\right\} \end{aligned}$$

Also

$$\begin{aligned} Var\left(U_f|\chi^2_{2n-2}, data\right) &= \tfrac{\hat{\theta}_0^2}{\left(\chi^2_{2n-2}\right)^2}\left\{4 + \left(\tfrac{n}{m}\right)^2\left[4 + \tilde{k}_2^2 2\left(2m-2\right)\right]\right\} \\ &= \tfrac{4\hat{\theta}_0}{\left(\chi^2_{2n-2}\right)^2}\left\{1 + \left(\tfrac{n}{m}\right)^2\left[1 + \tilde{k}_2^2\left(m-1\right)\right]\right\} \end{aligned}$$

The unconditional variance can be obtained by making use of the fact that

$$Var\left(U_f|data\right) = E_{\chi^2_{2n-2}}\left[Var\left(U_f|\chi^2_{2n-2}, data\right)\right] + Var_{\chi^2_{2n-2}}\left[E\left(U_f|\chi^2_{2n-2}, data\right)\right]$$

Now

$$E\left[\frac{1}{\left(\chi^2_{2n-2}\right)^2}\right] = \frac{1}{\left(2n-4\right)\left(2n-6\right)} = \frac{1}{4\left(n-2\right)\left(n-3\right)}$$

Therefore

$$E_{\chi^2_{2n-2}}\left[Var\left(U_f|\chi^2_{2n-2}, data\right)\right] = \frac{4\hat{\theta}_0}{\left(2n-4\right)\left(2n-6\right)}\left\{1 + \left(\frac{n}{m}\right)^2\left[1 + \tilde{k}_2^2\left(m-1\right)\right]\right\} \tag{C.1}$$

Also

$$\begin{aligned} Var\left(\tfrac{1}{\chi^2_{2n-2}}\right) &= E\left[\tfrac{1}{\left(\chi^2_{2n-2}\right)^2}\right] - E\left[\tfrac{1}{\chi^2_{2n-2}}\right]^2 \\ &= \tfrac{1}{(2n-4)(2n=6)} - \tfrac{1}{(2n-4)^2} = \tfrac{1}{4(n-2)^2(n-3)} \end{aligned}$$

and therefore

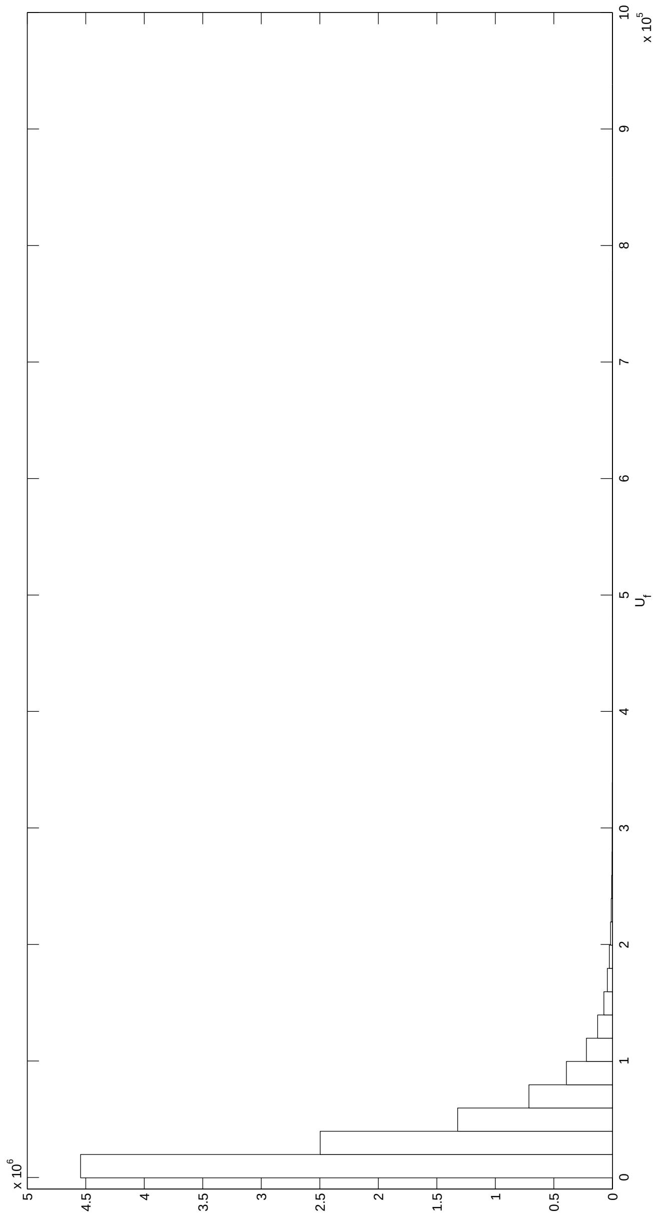
$$Var_{\chi^2_{2n-2}}\left[E\left(U_f|\chi^2_{2n-2}, data\right)\right] = \frac{\hat{\theta}_0}{4\left(n-2\right)^2\left(n-3\right)}\left\{2 - \frac{n}{m}\left[2 - \tilde{k}_2\left(2m-2\right)\right]\right\}^2 \tag{C.2}$$

From Equations C.1 and C.2 it follows that

$$\begin{aligned} Var\left(U_f|data\right) &= \tfrac{\hat{\theta}_0^2}{(n-2)(n-3)}\left\{1 + \left(\tfrac{n}{m}\right)^2\left[1 + \tilde{k}_2^2\left(m-1\right)\right]\right\} + \tfrac{\hat{\theta}_0^2}{(n-2)^2(n-3)}\left\{1 - \tilde{k}_2\left(m-1\right)\right\}^2 \\ &= \tfrac{\hat{\theta}_0^2}{(n-2)(n-3)}\left\{1 + \left(\tfrac{n}{m}\right)^2\left[1 + \tilde{k}_2^2\left(m-1\right)\right] + \tfrac{1}{(n-2)}\left\{1 - \tfrac{n}{m}\left[1 - \tilde{k}_2\left(m-1\right)\right]\right\}^2\right\} \end{aligned}$$

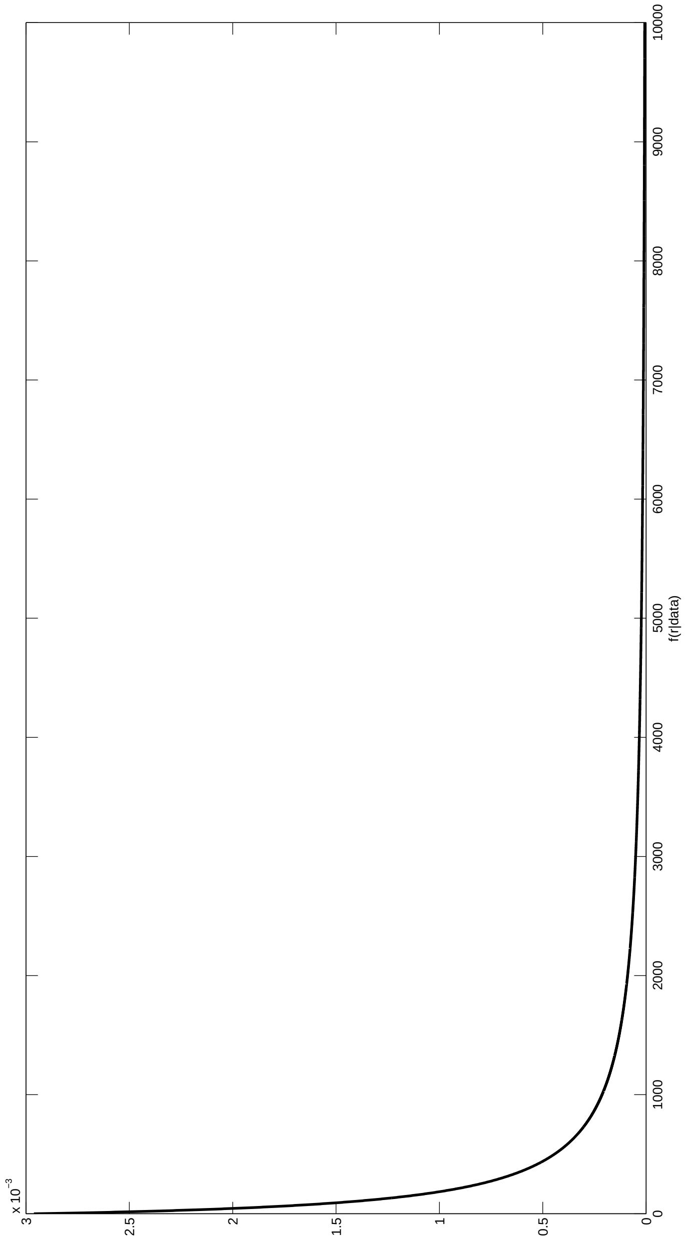
Figures

Figure 1: Predictive Density of U_f for $n = 19$ and $m = 2$



$$E\left(U_f|data\right) = 33554$$

Figure 2: Predictive Distribution of the "Run-length" $f\left(r|data\right)$ for $n = 19$ and $m = 2$



$$E\left(r|data\right) = 3959.2,\ Median\left(r|data\right) = 603.92,\ Var\left(r|data\right) = 1.7040e^{8}$$

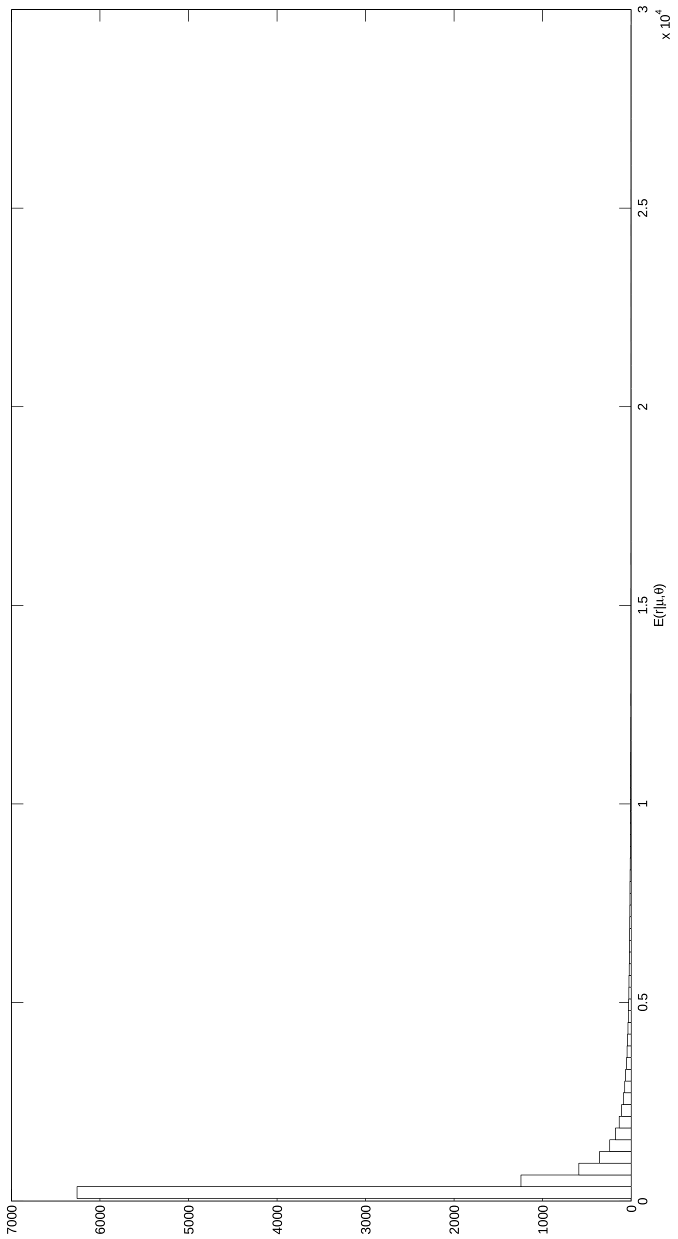
$$95\%\ Equal - tail\ Interval = \left(9.85; 37522\right),\ Length = 37512.15$$

$$95\%\ HPD\ Interval = \left(0; 18500\right),\ Length = 18500$$

$$Geometric\ Mean\left(r\right) = 1082.6$$

$$Harmonic\ Mean\left(r\right) = 366.65$$

Figure 3: Distribution of the Average "Run-length"



$$Mean\left[E\left(r|\mu,\theta\right)\right] = 5980.3$$

$$Median\left[E\left(r|\mu,\theta\right)\right] = 971.82$$