Bayesian Control Charts for the Two-parameter Exponential Distribution

R. van Zyl[1], A.J. van der Merwe[2]

[1]Quintiles International, ruaanvz@gmail.com

[2]University of the Free State

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Abstract

By using data that are the mileages for some military personnel carriers that failed in service given by Grubbs (1971) and Krishnamoorthy and Mathew (2009) a Bayesian procedure is applied to obtain control limits for the location and scale parameters, as well as for a one-sided upper tolerance limit in the case of the two-parameter exponential distribution. An advantage of the upper tolerance limit is that it monitors the location and scale parameter at the same time. By using Jeffreys' non-informative prior, the predictive distributions of future maximum likelihood estimators of the location and scale parameters are derived analytically. The predictive distributions are used to determine the distribution of the "run-length" and expected "run-length". This paper illustrates the flexibility and unique features of the Bayesian simulation method.

Keywords: Jeffreys' prior, two-parameter exponential, Bayesian procedure, run-length, control chart

1 Introduction

The two-parameter exponential distribution plays an important role in engineering, life testing and medical sciences. In these studies where the data are positively skewed, the exponential distribution is as important as the normal distribution is in sampling theory and agricultural statistics. Researchers have studied various aspects of estimation and inference for the two-parameter exponential distribution using either the frequentist approach or the Bayesian procedure.

However, while parameter estimation and hypothesis testing related to the two-parameter exponential distribution are well documented in the literature, the research on control charts has received little attention. Ramalhoto and Morais (1999) developed a control chart for monitoring the scale parameter while Sürücü and Sazak (2009) presented a control chart scheme in which moments are used. Mukherjee, McCracken, and Chakraborti (2014) on the other hand proposed several control charts and monitoring schemes for the location and the scale parameters of the two-parameter exponential distribution.

In this paper control charts for the location and scale parameters as well as for a one-sided upper tolerance limit will be developed by deriving their predictive distributions and using a Bayesian procedure.

Bayarri and García-Donato (2005) give the following reasons for recommending a Bayesian analysis:

- Control charts are based on future observations and Bayesian methods are very natural for prediction.
- Uncertainty in the estimation of the unknown parameters is adequately handled.
- Implementation with complicated models and in a sequential scenario poses no methodological difficulty, the numerical difficulties are easily handled via Monte Carlo methods;
- Objective Bayesian analysis is possible without introduction of external information other than the model, but any kind of prior information can be incorporated into the analysis, if desired.

Krishnamoorthy and Mathew (2009) and Hahn and Meeker (1991) defined a tolerance interval as an interval that is constructed in such a way that it will contain a specified proportion or more of the population with a certain degree of confidence. The proportion is also called the content of the tolerance interval. As opposed to confidence intervals that give information on unknown population parameters, a one-sided upper tolerance limit for example provides information about a quantile of the population.

2 Preliminary and Statistical Results

In this section the same notation will be used as given in Krishnamoorthy and Mathew (2009).

The two-parameter exponential distribution has the probability density function

$$f(x;\mu,\theta)=\frac{1}{\theta}\exp\left\{-\frac{(x-\mu)}{\theta}\right\}\qquad x>\mu,\ \mu>0,\ \theta>0$$

where μ is the location parameter and θ the scale parameter.

Let $X_1, X_2, \ldots, X_n$ be a sample of n observations from the two-parameter exponential distribution. The maximum likelihood estimators for μ and θ are given by

$$\hat{\mu}=X_{(1)}$$

and

$$\hat{\theta}=\frac{1}{n}\sum_{i=1}^{n}\left(X_i-X_{(1)}\right)=\bar{X}-X_{(1)}$$

where $X_{(1)}$ is the minimum or the first order statistic of the sample. It is well known (see Johnson and Kotz (1970); Lawless (1982); Krishnamoorthy and Mathew (2009)) that $\hat{\mu}$ and $\hat{\theta}$ are independently distributed with

$$\frac{(\hat{\mu}-\mu)}{\theta}\sim\frac{\chi^2_2}{2n}\text{ and }\frac{\hat{\theta}}{\theta}\sim\frac{\chi^2_{2n-2}}{2n}. \tag{2.1}$$

3 Bayesian Procedure

If a sample of n observations are drawn from the two-parameter exponential distribution, then the likelihood function is given by

$$L(\mu,\theta|data)=\left(\frac{1}{\theta}\right)^n\exp\left\{-\frac{1}{\theta}\sum_{i=1}^{n}(x_i-\mu)\right\}.$$

As prior the Jeffreys' prior

$$p(\mu,\theta)\propto\theta^{-1}$$

will be used.

The following theorems can now be proved.

Theorem 3.1. *The joint posterior distribution of μ and θ is*

$$\begin{aligned}p(\theta,\mu|data) &\propto p(\mu,\theta)\,L(\mu,\theta|data)\\ &= K_1\left(\tfrac{1}{\theta}\right)^{n+1}\exp\left\{-\tfrac{n}{\theta}(\bar{x}-\mu)\right\}\qquad 0<\mu<x_{(1)},\ 0<\theta<\infty\end{aligned} \tag{3.1}$$

where

$$K_1=\frac{n^n}{\Gamma(n-1)}\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1}-\left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1}.$$

Proof. The proof is given in Appendix A. □

Theorem 3.2. *The posterior distribution of* μ *is*

$$\begin{aligned} p(\mu|data) &= \int_0^\infty p(\theta,\mu|data)\,d\theta \\ &= (n-1)\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1} - \left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1}(\bar{x}-\mu)^{-n} \qquad 0<\mu<x_{(1)}. \end{aligned} \tag{3.2}$$

Proof. The proof is given in Appendix B. □

Theorem 3.3. *The posterior distribution of* θ *is*

$$\begin{aligned} p(\theta|data) &= \int_0^\infty p(\theta,\mu|data)\,d\mu \\ &= K_1\frac{1}{n}\left(\frac{1}{\theta}\right)^n\left\{\exp\left(-\frac{n\hat{\theta}}{\theta}\right) - \exp\left(-\frac{n\bar{x}}{\theta}\right)\right\} \qquad 0<\theta<\infty \end{aligned}. \tag{3.3}$$

Proof. The proof follows easily from the proof of Theorem 3.1. □

Theorem 3.4. *The conditional posterior distribution of* μ *given* θ *is*

$$\begin{aligned} p(\mu|\theta,data) &= \frac{p(\theta,\mu|data)}{p(\theta|data)} \\ &= K_2\exp\left(\frac{n\mu}{\theta}\right) \end{aligned} \tag{3.4}$$

where

$$K_2 = \frac{n}{\theta}\left\{\exp\left(\frac{nx_{(1)}}{\theta}\right) - 1\right\}^{-1}.$$

Proof. The proof follows easily from the proof of Theorem 3.1. □

Theorem 3.5. *The conditional posterior distribution of* θ *given* μ *is*

$$\begin{aligned} p(\theta|\mu,data) &= \frac{p(\theta,\mu|data)}{p(\mu|data)} \\ &= K_3\left(\frac{1}{\theta}\right)^{n+1}\exp\left\{-\frac{n}{\theta}(\bar{x}-\mu)\right\} \qquad 0<\theta<\infty \end{aligned} \tag{3.5}$$

where

$$K_3 = \frac{\{n(\bar{x}-\mu)\}^n}{\Gamma(n)}.$$

Proof. The proof is given in Appendix C. □

4 The Predictive Distributions of Future Sample Location and Scale Maximum Likelihood Estimators, $\hat{\mu}_f$ and $\hat{\theta}_f$

Consider a future sample of m observations from the two-parameter exponential population: $X_{1f}, X_{2f}, \ldots, X_{mf}$. The future sample mean is defined as $\bar{X}_f = \frac{1}{m}\sum_{j=1}^{m} X_{jf}$. The smallest value in the sample is denoted by $\hat{\mu}_f$ and $\hat{\theta}_f = \bar{X}_f - \hat{\mu}_f$. To obtain control charts for $\hat{\mu}_f$ and $\hat{\theta}_f$ their predictive distributions must first be derived.

The following theorems can now be proved.

Theorem 4.1. *The predictive distribution of a future sample location maximum likelihood estimator, $\hat{\mu}_f$, is given by*

$$f\left(\hat{\mu}_f|data\right) = \begin{cases} K^*\left\{\left[\frac{1}{n(\bar{x}-\hat{\mu}_f)}\right]^n - \left[\frac{1}{m\hat{\mu}_f+n\bar{x}}\right]^n\right\} & 0 < \mu_f < x_{(1)} \\ K^*\left\{\left[\frac{1}{m\left(\hat{\mu}_f - x_{(1)}\right)+n\hat{\theta}}\right]^n - \left[\frac{1}{m\hat{\mu}_f+n\bar{x}}\right]^n\right\} & x_{(1)} < \hat{\mu}_f < \infty \end{cases} \tag{4.1}$$

where

$$K^* = \frac{n^n\left(n-1\right)m}{\left(n+m\right)\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1} - \left(\frac{1}{\bar{x}}\right)^{n-1}\right\}}.$$

Proof. The proof is given in Appendix D. □

Theorem 4.2. *The mean of $\hat{\mu}_f$ is given by*

$$E\left(\hat{\mu}_f|data\right) = \bar{x} - \tilde{K}L\left(1-a\right)$$

and the variance by

$$Var\left(\hat{\mu}_f|data\right) = \left\{\frac{n^3}{m^2\left(n-1\right)^2\left(n-2\right)} + \left(1-a\right)^2\right\}\tilde{K}M - \left(1-a\right)^2\tilde{K}^2L^2$$

where

$$a = \frac{n}{m\left(n-1\right)},$$

$$\tilde{K} = \left(n-1\right)\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1} - \left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1},$$

$$L = \left(\frac{1}{n-2}\right)\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-2} - \left(\frac{1}{\bar{x}}\right)^{n-2}\right\}$$

and

$$M = \left(\frac{1}{n-3}\right)\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-3} - \left(\frac{1}{\bar{x}}\right)^{n-3}\right\}.$$

Proof. The proof is given in Appendix E. □

Theorem 4.3. *The predictive distribution of a future sample scale maximum likelihood estimator, $\hat{\theta}_f$, is given by*

$$f\left(\hat{\theta}_f|data\right) = m^{m-1}n^{n-1}\frac{\Gamma(m+n-2)}{\Gamma(m-1)\Gamma(n-1)}\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1}-\left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1}\hat{\theta}_f^{m-2}\times \left\{\left(\frac{1}{m\hat{\theta}_f+n\hat{\theta}}\right)^{m+n-2}-\left(\frac{1}{m\hat{\theta}_f+n\bar{x}}\right)^{m+n-2}\right\} \quad \hat{\theta}_f>0. \tag{4.2}$$

Proof. The proof is given in Appendix F. □

Theorem 4.4. *The mean and variance of $\hat{\theta}_f$ is given by*

$$E\left(\hat{\theta}_f|data\right) = \frac{n(m-1)}{m(n-1)}\tilde{K}L$$

and variance

$$Var\left(\hat{\theta}_f|data\right) = \frac{n^2(m-1)}{m(n-1)}\left\{\frac{\tilde{K}M}{n-2}-\frac{(m-1)}{m(n-1)}\tilde{K}^2L^2\right\}.$$

Proof. The proof is given in Appendix G. □

5 Example

The following data is given in Grubbs (1971) as well as in Krishnamoorthy and Mathew (2009). The failure mileages given in Table 5.1 fit a two-parameter exponential distribution.

Table 5.1: Failure Mileages of 19 Military Carriers

162	200	271	302	393	508	539	629	706	777
884	1008	1101	1182	1463	1603	1984	2355	2880	

For this data, the maximum likelihood estimates are $\hat{\mu} = x_{(1)} = 162$, $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - x_{(1)}\right) = \bar{x} - x_{(1)} = 835.21$ and $n = 19$.

As mentioned in the introductory section, the aim of this article is to obtain control charts for location and scale maximum likelihood estimates as well as for a one-sided upper tolerance limit.

Therefore in the next section a control chart for a future location maximum likelihood estimate will be developed.

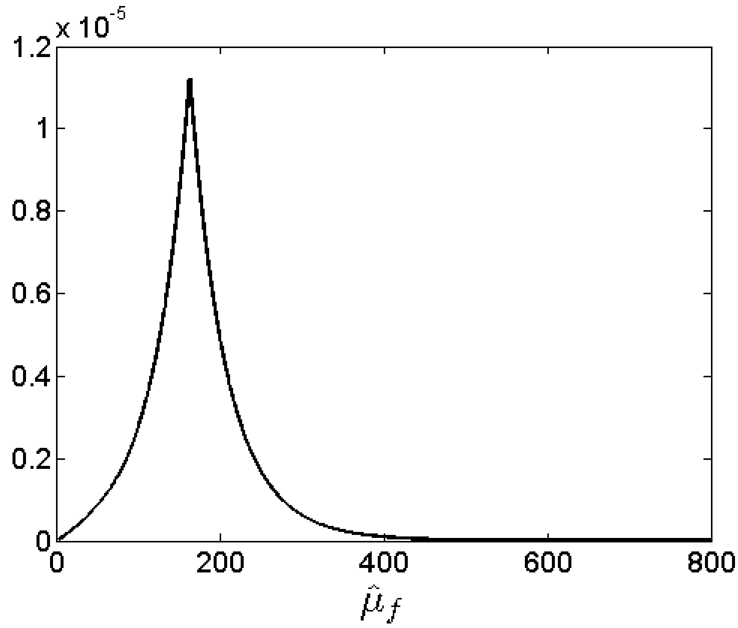
6 Control Chart for $\hat{\mu}_f$

It is well known that statistical quality control is actually implemented in two phases. In Phase I the primary interest is to assess process stability. The practitioner must therefore be sure that the process is in statistical control before control limits can be determined for online monitoring of the process in Phase II.

By using the predictive distribution (defined in Equation 4.1) a Bayesian procedure will be developed in Phase II to obtain a control chart for $\hat{\mu}_f$. Assuming that the process remains stable, the predictive distribution can be used to derive the distribution of the "run-length" and average "run-length".

For the example given in Table 5.1 (failure mileage data) the predictive distribution, $f(\hat{\mu}_f|data)$ for $m = 19$ future data is illustrated in Figure 6.1.

Figure 6.1: Distribution of $\hat{\mu}_f$, $n = 19$, $m = 19$



$Mean(\hat{\mu}_f) = 168.78$, $Median(\hat{\mu}_f) = 163.91$, $Mode(\hat{\mu}_f) = 162$, $Var(\hat{\mu}_f) = 3888.7$
$95\%\ Interval(\hat{\mu}_f) = (55.18; 317.45)$

$99.73\%\ Interval(\hat{\mu}_f) = (13.527; 489.52)$

From Figure 6.1 it follows that for a 99.73% two-sided control chart the lower control limit is $LCL = 13.527$ and the upper control limit is $UCL = 489.52$.

Let $R(\beta)$ represents those values of $\hat{\mu}_f$ that are smaller than LCL and larger than UCL.

The "run-length" is defined as the number of future $\hat{\mu}_f$ values (r) until the control chart signals for the first time (Note that r does not include the $\hat{\mu}_f$ value when the control chart signals). Given μ and θ and a stable Phase I process, the distribution of the "run-length" r is geometric with parameter

$$\psi(\mu,\theta) = \int_{R(\beta)} f(\hat{\mu}_f|\mu,\theta)\, d\hat{\mu}_f$$

where

$$f(\hat{\mu}_f|\mu,\theta) = \frac{m}{\theta}\exp\left\{-\frac{m}{\theta}(\hat{\mu}_f - \mu)\right\} \qquad \hat{\mu}_f > \mu$$

i.e., the distribution of $\hat{\mu}_f$ given that of μ and θ are known. See also Equation 2.1. The values of μ and θ are however unknown and the uncertainty of these parameter values are described by their joint posterior distribution $p(\theta,\mu|data)$ given in Equation 3.1.

By simulating μ and θ from $p(\theta,\mu|data) = p(\theta|\mu,data)\,p(\mu|data)$ the probability density function of $f(\hat{\mu}_f|\mu,\theta)$ as well as the parameter $\psi(\mu,\theta)$ can be obtained. This must be done for each future sample. In other words, for each future sample μ and θ must first be simulated from $p(\theta,\mu|data)$ and then $\psi(\mu,\theta)$

calculated. Therefore, by simulating all possible combinations of μ and θ from their joint posterior distribution a large number of $\psi(\mu, \theta)$ values can be obtained. Also, a large number of geometric distributions, i.e., a large number of "run-length" distributions each with a different parameter value $(\psi(\mu_1, \theta_1), \psi(\mu_2, \theta_2), \ldots, \psi(\mu_m, \theta_m))$ can be obtained.

As mentioned the "run-length" r for given μ and θ is geometrically distributed with mean

$$E(r|\mu, \theta) = \frac{1 - \psi(\mu, \theta)}{\psi(\mu, \theta)}$$

and variance

$$Var(r|\mu, \theta) = \frac{1 - \psi(\mu, \theta)}{\psi^2(\mu, \theta)}.$$

The unconditional moments, $E(r|data)$, $E(r^2|data)$ and $Var(r|data)$ can therefore be obtained by simulation or numerical integration. For further details refer to Menzefricke (2002, 2007, 2010a,b).

The mean of the predictive distribution of the "run-length" for the 99.73% two-sided control limits is $E(r|data) = 37526$, much larger than the 370 that one would have expected if $\beta = 0.0027$. The reason for this large average run-length is the small sample size and large variation in the data. The median run-length = 1450. Define $\tilde{\psi}(\mu, \theta) = \frac{1}{m}\sum_{i=1}^{m}\psi(\mu_i, \theta_i)$. From Menzefricke (2002) it follows that if $m \to \infty$, then $\tilde{\psi}(\mu, \theta) \to \beta$ and the harmonic mean of $r = \frac{1}{\beta}$. For $\beta = 0.0027$ the harmonic mean would therefore be 370.

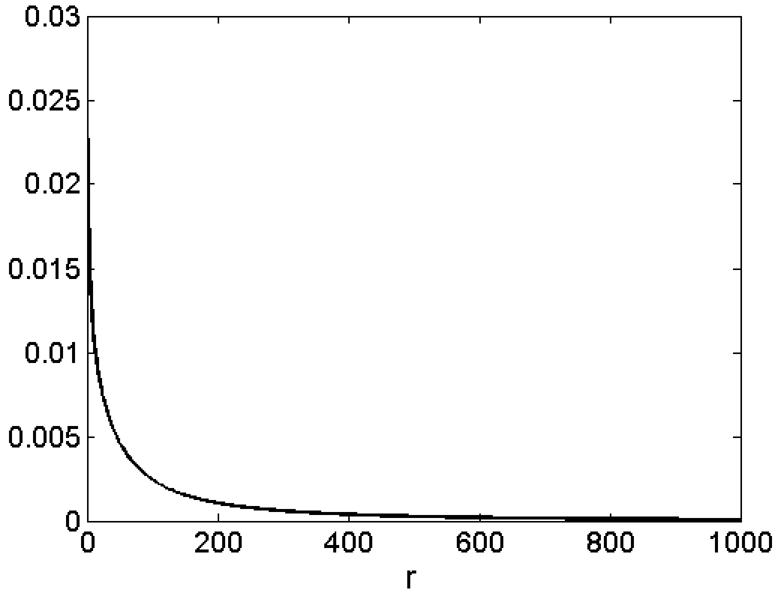
In Table 6.1 the average "run-length" for different values of β are given.

Table 6.1: β Values and Corresponding Average Run-length

β	0.0027	0.003	0.005	0.007	0.009	0.01	0.015	0.02	0.025	0.0258	0.03
$E(r\|data)$	37526	32610	10231	4655	2807	2313	987.6	594.2	399.7	369.67	280.1

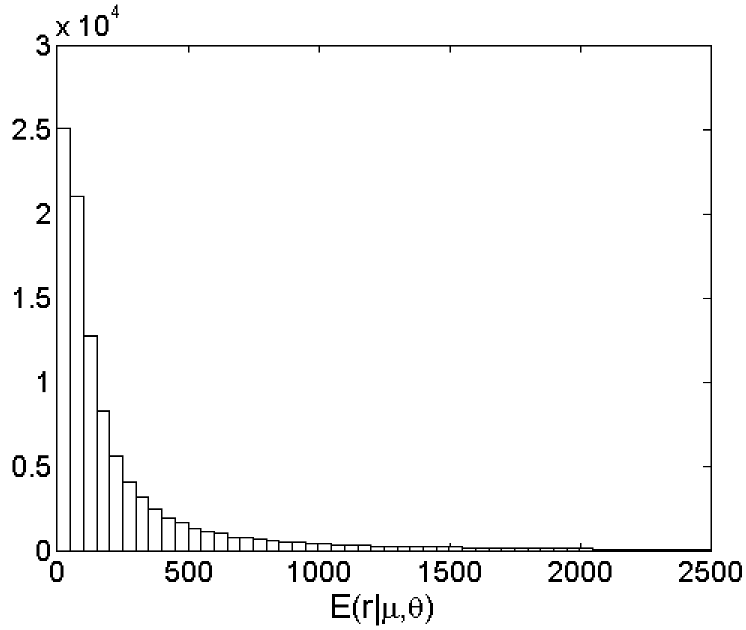
In Figure 6.2 the distribution of the "run-length" for $n = 19$, $m = 19$ and $\beta = 0.0258$ is illustrated and in Figure 6.3 the histogram of the expected "run-length" is given.

Figure 6.2: Run-length, $n = 19$, $m = 19$, $\beta = 0.0258$



$E(r|data) = 369.67$, $Median(r|data) = 71.22$, $Var(r|data) = 6.5275 \times 10^6$
$95\%\ interval\,(r|data) = (0; 1242.2)$

Figure 6.3: Expected Run-length, $n = 19$, $m = 19$, $\beta = 0.0258$

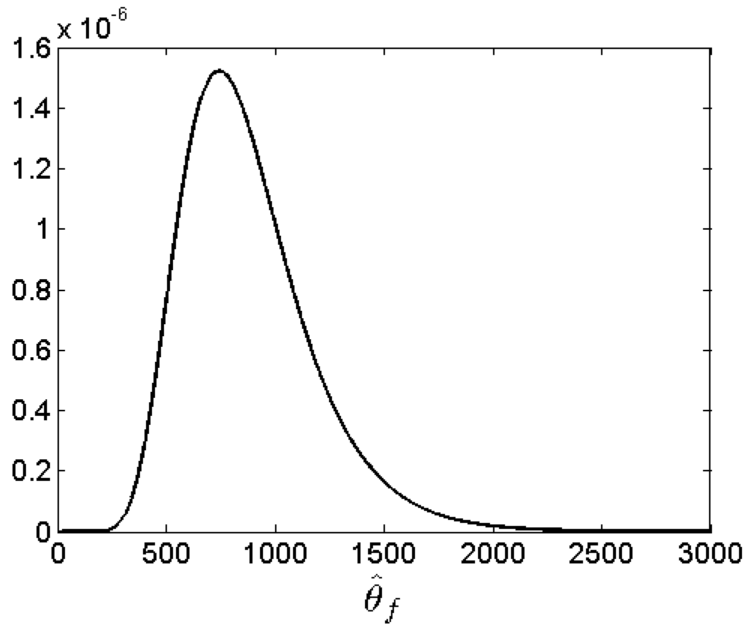


$Mean = 36.41$, $Median = 111.93$, $Var = 3.7157 \times 10^6$, $Mean\left(\psi\right) = 0.0261$
$95\%\ interval = (0; 1194.3)$

7 Control Chart for $\hat{\theta}_f$

In this section a control chart for $\hat{\theta}_f$, a future scale maximum likelihood estimator, will be developed. The predictive distribution $f\left(\hat{\theta}_f|data\right)$ given in Equation 4.2 is displayed in Figure 7.1 for the example previously given and $m = 19$.

Figure 7.1: Predictive Distribution of $\hat{\theta}_f$, $n = 19$, $m = 19$



$Mean\left(\hat{\theta}_f\right) = 876.73$, $Median\left(\hat{\theta}_f\right) = 829.23$, $Mode\left(\hat{\theta}_f\right) = 743.34$

For a 99.73% two-sided control chart the lower control limit is LCL = 297.5 and the upper control limit is UCL = 2278. $\tilde{R}(\beta)$ represents those values of $\hat{\theta}_f$ that are smaller than LCL and larger than UCL. Given θ and a stable Phase I process, the distribution of the "run-length" is geometric with parameter

$$\psi(\theta) = \int_{\tilde{R}(\beta)} f\left(\hat{\theta}_f|\theta\right) d\hat{\theta}_f$$

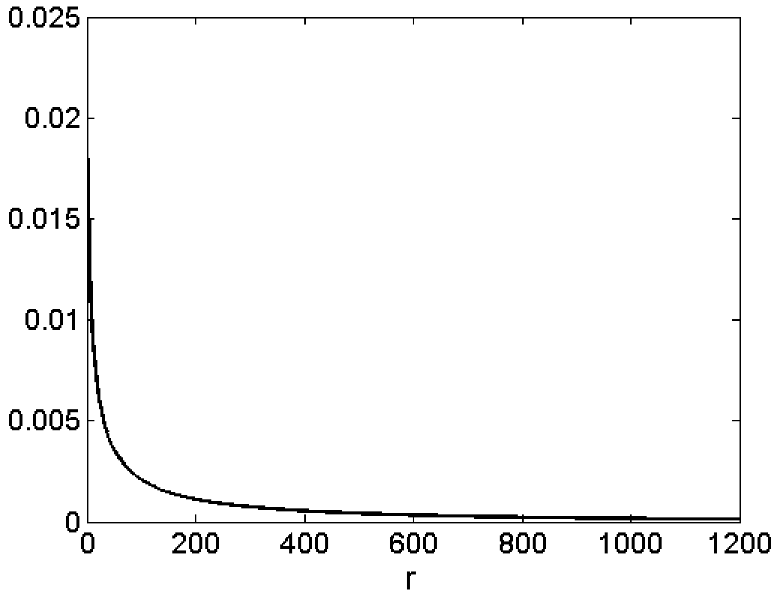
where $f\left(\hat{\theta}_f|\theta\right)$ is defined in Equation 2.1. As before the value of θ is unknown, but can be simulated from the posterior distribution $p(\theta|data)$ given in Equation 3.3 or equivalently by first simulating μ from $p(\mu|data)$ and then θ from $p(\theta|\mu, data)$. By simulating θ the probability density function $f\left(\hat{\theta}_f|\theta\right)$ Equation 3.3 as well as the parameter $\psi(\theta)$ can be obtained. As mentioned in Section 6, this must be done for each future sample. The mean of the predictive distribution of the "run-length" for the 99.73% two-sided control limits is $E(r|data) = 8188.6$. As in the case of $\hat{\mu}_f$, this is much larger than the 370 that one would have expected if $\beta = 0.0027$. In Table 7.1 the average run-length versus probabilities β are given.

Table 7.1: β Values and Corresponding Average Run-length

β	0.0027	0.003	0.005	0.007	0.009	0.01	0.015	0.018	0.02	0.025
$E(r\|data)$	10939.2	9029.8	3517.7	1913	1238.2	1010.7	512.1	372.4	311.2	211.8

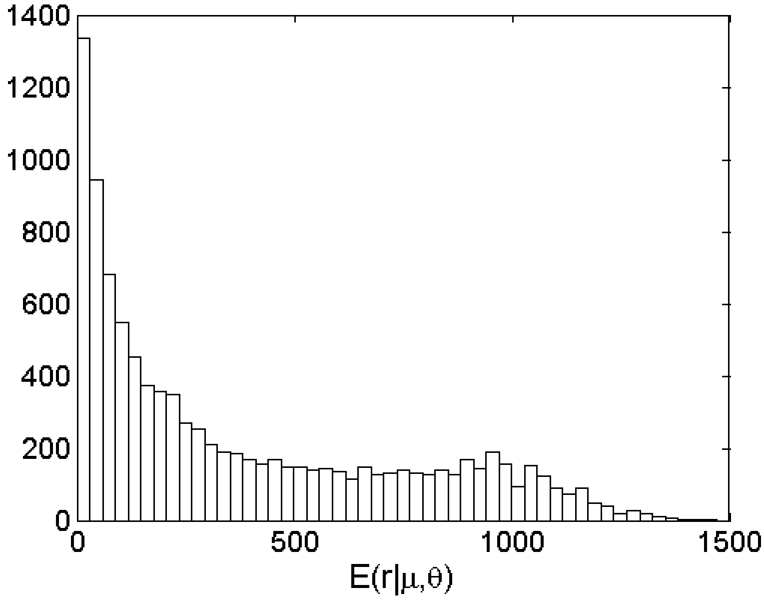
In Figure 7.2 the distribution of the "run-length" for $n = 19$, $m = 19$ and $\beta = 0.018$ is illustrated and in Figure 7.3 the histogram of the expected "run-length" is given.

Figure 7.2: Run Length, $n = 19$, $m = 19$, $\beta = 0.018$



$E(r|data) = 372.4$, $Median(r|data) = 127.4$, $Var(r|data) = 3.9515 \times 10^5$
$95\%\ interval(r|data) = (0; 1602.5)$

Figure 7.3: Expected Run-length, $n = 19$, $m = 19$, $\beta = 0.018$



$Mean = 372.4$, $Median = 238.38$, $Var = 1.2698 \times 10^5$, $Mean\,(\psi) = 0.0177$
$95\%\ interval = (7.809; 1096.7)$

8 A Bayesian Control Chart for a One-sided Upper Tolerance Limit

As mentioned in the introduction, a confidence interval for a quantile is called a tolerance interval. A one-sided upper tolerance limit is therefore a quantile of a quantile. It can easily be shown that the p quantile of a two-parameter exponential distribution is given by $q_p = \mu - \theta \ln (1 - p)$. By replacing the parameters by their generalized pivotal quantities (GPQs), Krishnamoorthy and Mathew (2009) showed that a GPQ for q_p can be obtained as

$$G_{q_p} = \hat{\mu} - \left[\frac{\chi_2^2 + 2n \ln (1-p)}{\chi_{2n-2}^2} \right] \hat{\theta}.$$

Let $E_{p;\alpha}$ denotes the α quantile of $E_p = \frac{\chi_2^2 + 2n \ln(1-p)}{\chi_{2n-2}^2}$, then

$$\hat{\mu} - E_{p;\alpha}\hat{\theta} = \hat{\mu} - \tilde{k}_2 \hat{\theta} \tag{8.1}$$

is a $1 - \alpha$ upper confidence limit for q_p, which means that $(p, 1 - \alpha)$ is an upper tolerance limit for the *exponential* (μ, θ) distribution.

An advantage of the upper tolerance limit is that it monitors the location and scale parameters of the two-parameter exponential distribution at the same time.

It is shown in Roy and Mathew (2005) and Krishnamoorthy and Mathew (2009) that the upper and lower tolerance limits are actually exact, which means that they have the correct frequentist coverage probabilities.

9 The Predictive Distribution of a Future Sample Upper Tolerance Limit

From Equation 8.1 it follows that a future sample upper tolerance limit is defined as

$$U_f = \hat{\mu}_f - \tilde{k}_2 \hat{\theta}_f \text{ where } \hat{\mu}_f > \mu \text{ and } \hat{\theta}_f > 0.$$

From Equation 2.1 it follows that

$$f\left(\hat{\mu}_f | \mu, \theta\right) = \left(\frac{m}{\theta}\right) \exp\left\{-\frac{m}{\theta}\left(\hat{\mu}_f - \mu\right)\right\} \; \hat{\mu}_f > \mu$$

which means that

$$f\left(U_f | \mu, \theta, \hat{\theta}_f\right) = \left(\frac{m}{\theta}\right) \exp\left\{-\frac{m}{\theta}\left[U_f - \left(\mu - \tilde{k}_2 \hat{\theta}_f\right)\right]\right\} \qquad U_f > \mu - \tilde{k}_2 \hat{\theta}_f \tag{9.1}$$

From Equation 9.1 it can be seen that the derivation of the unconditional predictive density function $f\left(U_f | data\right)$ will be quite complicated. An approximation of the density function can however be obtained by using the following Monte Carlo simulation procedure:

1. Simulate μ and θ from $p\left(\theta, \mu | data\right)$. This can be achieved by first simulating μ from $p\left(\mu | data\right)$ defined in Equation 3.2 and then θ from $p\left(\theta | \mu, data\right)$ defined in Equation 3.5.
2. For given θ, simulate $\hat{\theta}_f$ from $\frac{\chi^2_{2m-2}}{2m}\theta$.
3. Substitute the simulated μ, θ, and $\hat{\theta}_f$ values in Equation 9.1 and draw the exponential distribution.

Repeat this procedure l times and obtain the average of the l simulated exponential density functions (Rao-Blackwell method) to obtain the unconditional predictive density $f\left(U_f | data\right)$.

Although (as mentioned) the derivation of the exact unconditional predictive density will be quite complicated the exact moments can be derived analytically. The following theorem can be proved:

Theorem 9.1. *The exact mean and variance of $U_f = \hat{\mu}_f - \tilde{k}_2 \hat{\theta}_f$, a future sample tolerance limit, is given by*

$$E\left(U_f | data\right) = \bar{x} + \tilde{K} L \left(aH - 1\right) \tag{9.2}$$

and

$$Var\left(U_f | data\right) = \frac{n^2}{m^2 (n-1)(n-2)} \left\{J + \frac{H^2}{n-1}\right\} \tilde{K} M + (1 - aH)^2 \left\{\tilde{K} M - \tilde{K}^2 L^2\right\} \tag{9.3}$$

where

$$H = \left\{1 - \tilde{k}_2 (m-1)\right\}$$

and

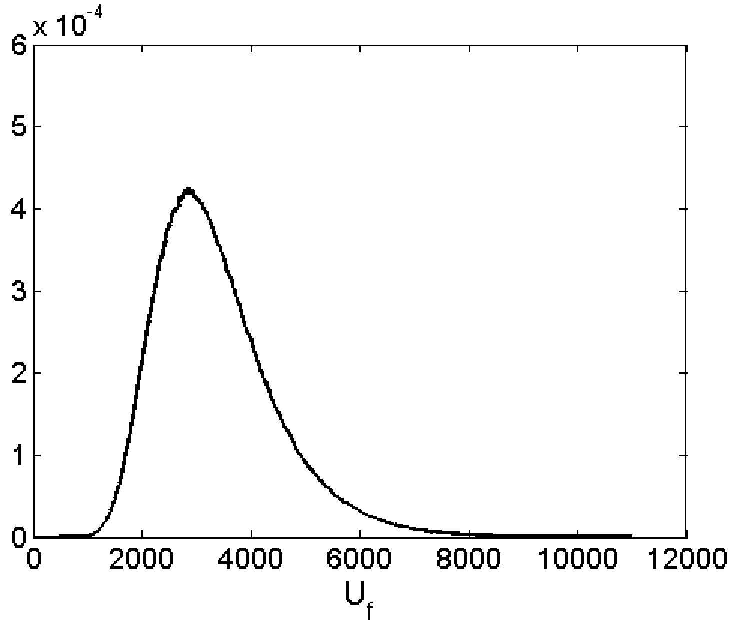
$$J = 1 + \tilde{k}_2^2 (m-1).$$

$\tilde{K}$, L, M and a are defined in Theorem 4.2.

Proof. The proof is given in Appendix H. □

For the failure mileage data given in Table 5.1, $\tilde{k}_2 = -3.6784$ if $m = 19$ and by using $10,000,000$ Monte Carlo simulations as described in (1), (2) and (3), the unconditional predictive density function can be obtained and is illustrated in Figure 9.1. Figure 9.1 is therefore the distribution of an upper future tolerance limit for the mileages of the next 19 military personnel carriers that will fail in service.

Figure 9.1: Predictive Density of $f\left(U_f|data\right)$



$Mean\left(U_f\right) = 3394.7$, $Mode\left(U_f\right) = 2900$, $Median\left(U_f\right) = 3211.5$, $Var\left(U_f\right) = 1.2317 \times 10^6$
95% $interval\left(U_f\right) = (1736.5; 6027)$

99.73% $interval\left(U_f\right) = (1249.05, 7973)$

As in the previous sections the predictive distribution can be used to derive the "run-length" and average "run-length". From Figure (9.1) it follows that for a 99.73% two-sided control chart the lower control limit is LCL = 1249.05 and the upper control limit is UCL = 7973. $R^*(\beta)$ therefore represents those values of U_f that are smaller than LCL and larger than UCL. As before, the "run-length" is defined as the number of future U_f values (r) until the control chart signals for the first time. Given μ and θ the distribution of the "run-length" r is geometric with parameter

$$\tilde{\psi}(\mu,\theta) = \int_{R^*(\beta)} f\left(U_f|\mu,\theta\right) dU_f$$

where $f\left(U_f|\mu,\theta\right)$ is the distribution of a future U given that μ and θ are known. As mentioned before the values of μ and θ are however unknown and the uncertainty of these parameter values are described by their joint posterior distribution $p\left(\mu,\theta|data\right)$.

By simulating μ and θ from $p\left(\mu,\theta|data\right)$, the probability density function $f\left(U_f|\mu,\theta\right)$ can be obtained from Equations 2.1 and 9.1 in the following way.

I. $f\left(U_f|\mu,\theta,\chi^2_{2m-2}\right) = \left(\frac{m}{\theta}\right)\exp\left\{-\frac{m}{\theta}\left[U_f - \left(\mu - \tilde{k}_2\frac{\chi^2_{2m-2}}{2m}\theta\right)\right]\right\}$.

II. The next step is to simulate $l^* = 100000$ χ^2_{2m-2} values to obtaine l^* exponential density functions for given μ and θ.

III. By averaging the l density functions (Rao-Blackwell method) $f(U_f|\mu,\theta)$ can be obtained and also $\tilde{\psi}(\mu,\theta)$.

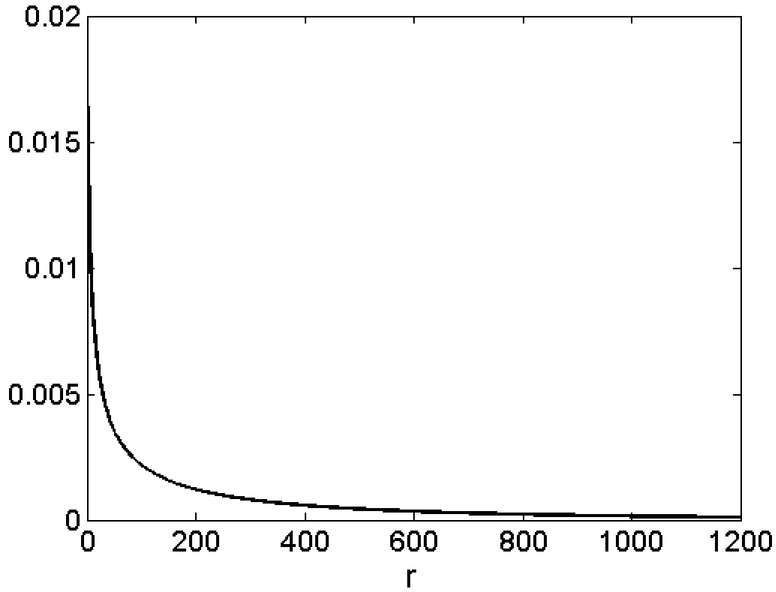
This must be done for each future sample. In other words, for each future sample μ and θ must be simulated from $p(\mu,\theta|data)$ and then the steps described in (I.), (II.) and (III.). The mean of the predictive distribution of the "run-length" for the 99.73% two-sided control limits is $E(r|data) = 1.1709 \times 10^{11}$, much larger than the 370 that would have been expected for $\beta = 0.0027$. In Table 9.1 the average "run-lengths" versus probabilities β are given.

Table 9.1: β Values and Corresponding Average Run-length

β	0.007	0.009	0.01	0.015	0.018	0.02	0.025
$E(r|data)$	10000	1240.2	1020.7	530.1	374.2	300.2	208.8

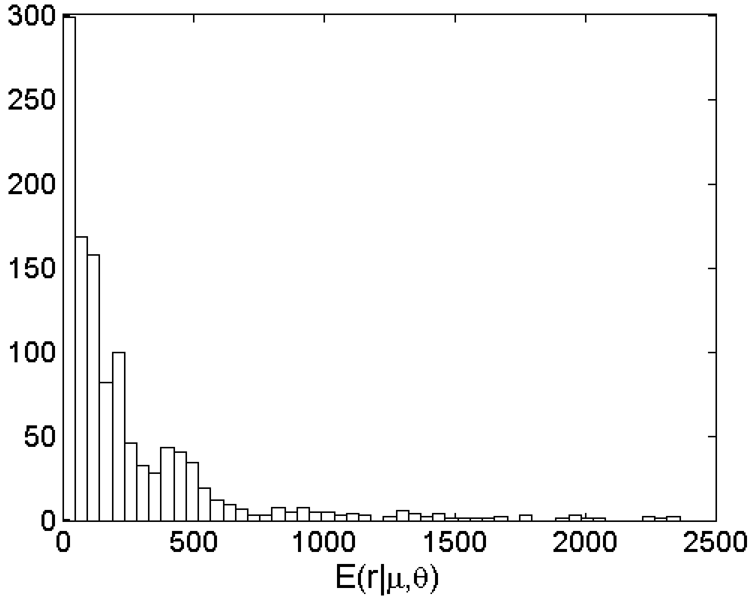
In Figure 9.2 the distribution of the "run-length" for $n = 19$, $m = 19$ and $\beta = 0.018$ is illustrated and Figure 9.3 the histogram of the expected "run-length" is given.

Figure 9.2: Distribution of Run-length when $\beta = 0.018$



$E(r|data) = 374.2$, $Median(r|data) = 132.1$, $Var(r|data) = 5.8236 \times 10^5$
$95\%\ interval(r|data) = (0; 1803.6)$

Figure 9.3: Expected Run-length when $\beta = 0.018$



$Mean = 374.2$, $Median = 248.03$, $Var = 2.0351 \times 10^5$
$95\%\ interval = (3.5541; 1463.2)$

10 Conclusion

This paper develops a Bayesian control chart for monitoring the scale parameter, location parameter and upper tolerance limit of a two-parameter exponential distribution. In the Bayesian approach prior knowledge about the unknown parameters is formally incorporated into the process of inference by assigning a prior distribution to the parameters. The information contained in the prior is combined with the likelihood function to obtain the posterior distribution. By using the posterior distribution the predictive distributions of $\hat{\mu}_f$, $\hat{\theta}_f$ and U_f can be obtained.

The theory and results described in this paper have been applied to the failure mileages for military carriers analyzed by Grubbs (1971) and Krishnamoorthy and Mathew (2009). The example illustrates the flexibility and unique features of the Bayesian simulation method for obtaining posterior distributions and "run-lengths" for $\hat{\mu}_f$, $\hat{\theta}_f$ and U_f.

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Mathematical Appendices

A Proof of Theorem 3.1

$$
\begin{aligned}
K_1^{-1} &= \int_0^\infty \int_0^{x_{(1)}} \left(\tfrac{1}{\theta}\right)^{n+1} \exp\left\{-\tfrac{n}{\theta}\left(\bar{x}-\mu\right)\right\} d\mu d\theta \\
&= \int_0^\infty \left(\tfrac{1}{\theta}\right)^{n+1} \exp\left\{-\tfrac{n}{\theta}\bar{x}\right\} \left[\int_0^{x_{(1)}} \exp\left(\tfrac{n\mu}{\theta}\right) d\mu\right] d\theta
\end{aligned}
$$

Since

$$
\int_0^{x_{(1)}} \exp\left(\frac{n\mu}{\theta}\right) d\mu = \frac{\theta}{n}\left\{\exp\left(\frac{nx_{(1)}}{\theta}\right) - 1\right\}
$$

it follows that

$$
\begin{aligned}
K_1^{-1} &= \tfrac{1}{n}\int_0^\infty \left(\tfrac{1}{\theta}\right)^{n} \exp\left\{-\tfrac{n\hat{\theta}}{\theta}\right\} d\theta - \tfrac{1}{n}\int_0^\infty \left(\tfrac{1}{\theta}\right)^{n} \exp\left\{-\tfrac{n\bar{x}}{\theta}\right\} d\theta \\
&= \left(\tfrac{1}{n}\right)^n \left(\tfrac{1}{\hat{\theta}}\right)^{n-1} \Gamma\left(n-1\right) - \left(\tfrac{1}{n}\right)^n \left(\tfrac{1}{\bar{x}}\right)^{n-1} \Gamma\left(n-1\right) \\
&= \left(\tfrac{1}{n}\right)^n \Gamma\left(n-1\right) \left\{\left(\tfrac{1}{\hat{\theta}}\right)^{n-1} - \left(\tfrac{1}{\bar{x}}\right)^{n-1}\right\}.
\end{aligned}
$$

Since $K_1 = \left(K_1^{-1}\right)^{-1}$ the theorem follows.

B Proof of Theorem 3.2

$$
\begin{aligned}
p\left(\mu|data\right) &= \int_0^\infty p\left(\mu,\theta|data\right) d\theta \\
&= K_1 \int_0^\infty \left(\tfrac{1}{\theta}\right)^{n+1} \exp\left\{-\tfrac{n}{\theta}\left(\bar{x}-\mu\right)\right\} d\theta \\
&= K_1 \left\{\tfrac{1}{n(\bar{x}-\mu)}\right\}^n \Gamma\left(n\right).
\end{aligned}
$$

By substituting K_1 the result follows.

C Proof of Theorem 3.5

Since

$$
\begin{aligned}
K_3^{-1} &= \int_0^\infty \left(\tfrac{1}{\theta}\right)^{n+1} \exp\left\{-\tfrac{n}{\theta}\left(\bar{x}-\mu\right)\right\} d\theta \\
&= \tfrac{\Gamma(n)}{\{n(\bar{x}-\mu)\}^n},
\end{aligned}
$$

the result follows.

D Proof of Theorem 4.1

From Equation 2.1 it follows that

$$\hat{\mu}_f|\mu,\theta \sim \frac{\chi_2^2}{2m}\theta + \mu \qquad \hat{\mu}_f > \mu.$$

Therefore

$$f\left(\hat{\mu}_f|\mu,\theta\right) = \left(\frac{m}{\theta}\right)\exp\left\{-\frac{m}{\theta}\left(\hat{\mu}_f - \mu\right)\right\} \qquad \hat{\mu}_f > \mu.$$

Further

$$f\left(\hat{\mu}|data\right) = \int\int f\left(\hat{\mu}|\mu,\theta\right)p\left(\theta|\mu,data\right)p\left(\mu|data\right)d\theta d\mu$$

where

$$p\left(\theta|\mu,data\right) = \frac{\left\{n\left(\bar{x}-\mu\right)\right\}^n}{\Gamma\left(n\right)}\left(\frac{1}{\theta}\right)^{n+1}\exp\left\{-\frac{n}{\theta}\left(\bar{x}-\mu\right)\right\} \qquad 0 < \theta < \infty$$

and

$$p\left(\mu|data\right) = \left(n-1\right)\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1} - \left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1}\left(\bar{x}-\mu\right)^{-n} \qquad 0 < \mu < x_{(1)}.$$

Now

$$\begin{aligned} f\left(\hat{\mu}_f|\mu,data\right) &= \int_0^\infty f\left(\hat{\mu}_f|\mu,\theta\right)p\left(\theta|\mu,data\right)d\theta \\ &= \tfrac{m\left\{n\left(\bar{x}-\mu\right)\right\}^n}{\Gamma(n)}\int_0^\infty \left(\tfrac{1}{\theta}\right)^{n+2}\exp\left\{-\tfrac{1}{\theta}\left[m\left(\hat{\mu}_f-\mu\right)+n\left(\bar{x}-\mu\right)\right]\right\}d\theta \\ &= \tfrac{n^{n+1}\left(\bar{x}-\mu\right)^n m}{\left[m\left(\hat{\mu}_f-\mu\right)+n\left(\bar{x}-\mu\right)\right]^{n+1}} \qquad \hat{\mu}_f > 0 \end{aligned}$$

and

$$\begin{aligned} f\left(\hat{\mu}_f|data\right) &= \int_0^{\hat{\mu}_f} f\left(\hat{\mu}_f,\mu|data\right)d\mu \qquad 0 < \hat{\mu}_f < x_{(1)} \\ &= \int_0^{x_{(1)}} f\left(\hat{\mu}_f,\mu|data\right)d\mu \qquad x_{(1)} < \hat{\mu}_f < \infty \end{aligned}$$

where

$$f\left(\hat{\mu}_f,\mu|data\right) = \frac{n^{n+1}\left(n-1\right)m}{\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1} - \left(\frac{1}{\bar{x}}\right)^{n-1}\right\}}\left\{\left(m\hat{\mu}_f + n\bar{x}\right) - \mu\left(n+m\right)\right\}^{-(n-1)}.$$

Therefore

$$\begin{aligned} f\left(\hat{\mu}_f|data\right) &= K^*\left\{\left[\tfrac{1}{n\left(\bar{x}-\hat{\mu}_f\right)}\right]^n - \left[\tfrac{1}{m\hat{\mu}_f+n\bar{x}}\right]^n\right\} \qquad 0 < \hat{\mu}_f < x_{(1)} \\ &= K^*\left\{\left[\tfrac{1}{m\left(\hat{\mu}_f-x_{(1)}\right)+n\hat{\theta}}\right]^n - \left[\tfrac{1}{m\hat{\mu}_f+n\bar{x}}\right]^n\right\} \qquad x_{(1)} < \hat{\mu}_f < \infty \end{aligned}$$

and

$$K^* = \frac{n^n\left(n-1\right)m}{\left(n+m\right)}\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1} - \left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1}.$$

E Proof of Theorem 4.2

Expected Value of $\hat{\mu}_f$

It follows from Equation 2.1 that

$$\hat{\mu}_f \sim \theta \frac{\chi_2^2}{2m} + \mu.$$

Therefore

$$E\left(\hat{\mu}_f|\mu,\theta\right) = \frac{\theta}{m} + \mu.$$

From Equation 3.1 it follows that

$$p\left(\theta|\mu, data\right) = \frac{\left\{n\left(\bar{x}-\mu\right)\right\}^n}{\Gamma\left(n\right)}\left(\frac{1}{\theta}\right)^{n+1} \exp\left\{-\frac{n}{\theta}\left(\bar{x}-\mu\right)\right\}$$

which means that

$$E\left(\theta|\mu, data\right) = \frac{n\left(\bar{x}-\mu\right)}{\left(n-1\right)}$$

and therefore

$$\begin{aligned} E\left(\hat{\mu}_f|\mu, data\right) &= \tfrac{n(\bar{x}-\mu)}{m(n-1)} + \mu \\ &= \left(1-a\right)\mu + a\bar{x} \end{aligned} \tag{E.1}$$

where

$$a = \frac{n}{m\left(n-1\right)}.$$

Also from Equation 3.2 it follows that

$$p\left(\mu|data\right) = \tilde{K}\left(\bar{x}-\mu\right)^{-n} \qquad 0 < \mu < x_{(1)}$$

where

$$\tilde{K} = \left(n-1\right)\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1} - \left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1}.$$

Now

$$E\left(\mu|data\right) = -E\left\{\left(\bar{x}-\mu\right)|data\right\} + \bar{x}$$

and

$$E\left\{\left(\bar{x}-\mu\right)|data\right\} = \frac{\tilde{K}}{\left(n-2\right)}\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-2} - \left(\frac{1}{\bar{x}}\right)^{n-2}\right\} = \tilde{K}L$$

which means that

$$E\left(\mu|data\right) = -\tilde{K}L + \bar{x}. \tag{E.2}$$

Substitute Equation E.2 in E.1 and the result follows as

$$E\left(\hat{\mu}_f|data\right) = \bar{x} - \tilde{K}L\left(1-a\right).$$

Variance of $\hat{\mu}_f$

From Equation 2.1 it also follows that

$$Var\left(\hat{\mu}_f|\mu, data\right) = \theta^2 \frac{4}{4m^2} = \frac{\theta^2}{m^2}.$$

Further

$$Var\left(\hat{\mu}_f|\mu, data\right) = E_{\theta|\mu}\left\{Var\left(\hat{\mu}_f|\mu,\theta\right)\right\} + Var_{\theta|\mu}\left\{E\left(\hat{\mu}_f|\mu,\theta\right)\right\}.$$

Since

$$E\left(\theta^2|\mu, data\right) = \frac{\left\{n\left(\bar{x}-\mu\right)\right\}^2}{\left(n-1\right)\left(n-2\right)}$$

and

$$Var\left(\theta|\mu, data\right) = \frac{\left\{n\left(\bar{x}-\mu\right)\right\}^2}{\left(n-1\right)^2\left(n-2\right)}$$

it follows that

$$\begin{aligned} Var\left(\hat{\mu}_f|\mu, data\right) &= \tfrac{1}{m^2}\tfrac{\{n(\bar{x}-\mu)\}^2}{(n-1)(n-2)} + \tfrac{1}{m^2}\tfrac{\{n(\bar{x}-\mu)\}^2}{(n-1)^2(n-2)} \\ &= \tfrac{n^3(\bar{x}-\mu)^2}{m^2(n-1)^2(n-2)}. \end{aligned}$$

Also

$$Var\left(\hat{\mu}_f|data\right) = E_{\mu}\left\{Var\left(\hat{\mu}_f|\mu, data\right)\right\} + Var_{\mu}\left\{E\left(\hat{\mu}_f|\mu, data\right)\right\}.$$

Since

$$E\left(\bar{x}-\mu\right)^2 = \tilde{K}M$$

where

$$M = \left(\frac{1}{n-3}\right)\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-3} - \left(\frac{1}{\bar{x}}\right)^{n-3}\right\}$$

it follows that

$$E_{\mu}\left\{Var\left(\hat{\mu}_f|\mu, data\right)\right\} = \frac{n^3}{m^2\left(n-1\right)^2\left(n-2\right)}\tilde{K}M.$$

Further

$$Var_{\mu}\left\{E\left(\hat{\mu}_f|\mu, data\right)\right\} = \left(1-a\right)^2 Var\left(\mu|data\right)$$

and

$$\begin{aligned} Var\left(\mu|data\right) &= E_{\mu}\left\{\mu - \left(\bar{x}-\tilde{K}L\right)\right\}^2 \\ &= E_{\mu}\left\{\left(\bar{x}-\mu\right)^2 - 2\left(\bar{x}-\mu\right)\tilde{K}L + \tilde{K}^2L^2\right\} \\ &= \tilde{K}M - 2\tilde{K}L\tilde{K}L + \tilde{K}^2L^2 = \tilde{K}M - \tilde{K}^2L^2. \end{aligned}$$

Therefore

$$Var_{\mu}\left\{E\left(\hat{\mu}_f|\mu, data\right)\right\} = \left(1-a\right)^2\left\{\tilde{K}M - \tilde{K}^2L^2\right\}$$

and

$$Var\left(\hat{\mu}_f|data\right) = \left\{\frac{n^3}{m^2\left(n-1\right)^2\left(n-2\right)} + \left(1-a\right)^2\right\}\tilde{K}M - \left(1-a\right)^2\tilde{K}^2L^2.$$

F Proof of Theorem 4.3

From Equation 2.1 it follows that

$$\hat{\theta}_f|\theta \sim \frac{\chi^2_{2m-2}}{2m}\theta \qquad \hat{\theta}_f > 0.$$

Therefore

$$f\left(\hat{\theta}_f|\theta\right) = \left(\frac{m}{\theta}\right)^{m-1} \frac{\left(\hat{\theta}_f\right)^{m-2} \exp\left(-\frac{m}{\theta}\hat{\theta}_f\right)}{\Gamma(m-1)} \qquad \hat{\theta}_f > 0.$$

The posterior distribution of θ is

$$p(\theta|data) = K_1 \left(\frac{1}{\theta}\right)^n \left\{\exp\left(-\frac{n}{\theta}\hat{\theta}\right) - \exp\left(-\frac{n}{\theta}\bar{x}\right)\right\}$$

where

$$K_1 = \frac{n^{n-1}}{\Gamma(n-1)} \left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1} - \left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1}$$

(see Theorem 3.3).

The unconditional predictive density function of $\hat{\theta}_f$ is therefore given by

$$\begin{aligned}
f\left(\hat{\theta}_f|data\right) &= \int_0^\infty f\left(\hat{\theta}_f|\theta\right) p(\theta|data)\, d\theta \\
&= \frac{m^{m-1}}{\Gamma(m-1)}\hat{\theta}_f K_1 \int_0^\infty \left(\frac{1}{\theta}\right)^{m+n-1} \left\{\exp\left[-\frac{1}{\theta}\left(m\hat{\theta}_f + n\hat{\theta}\right)\right] - \exp\left[-\frac{1}{\theta}\left(m\hat{\theta}_f + n\bar{x}\right)\right]\right\} d\theta \\
&= m^{m-1}n^{n-1}\frac{\Gamma(m+n-2)}{\Gamma(m-1)\Gamma(n-1)} \left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1} - \left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1} \left(\hat{\theta}_f\right)^{m-2} \\
&\quad \times \left\{\left(\frac{1}{m\hat{\theta}_f + n\hat{\theta}}\right)^{m+n-2} - \left(\frac{1}{m\hat{\theta}_f + n\bar{x}}\right)^{m+n-2}\right\} \qquad \hat{\theta}_f > 0.
\end{aligned}$$

G Proof of Theorem 4.4

Expected Value of $\hat{\theta}_f$

From Equation 2.1 it follows that

$$\hat{\theta}_f|\mu,\theta \sim \frac{\chi^2_{2m-2}}{2m}\theta.$$

Therefore

$$E\left(\hat{\theta}_f|\mu,\theta\right) = \frac{(m-1)}{m}\theta$$

and

$$Var\left(\hat{\theta}_f|\mu,\theta\right) = \frac{(m-1)}{m^2}\theta^2.$$

By using $p(\theta|\mu, data)$ (given in Equation 3.5) it follows that

$$E(\theta|\mu, data) = \frac{n(\bar{x}-\mu)}{(n-1)}$$

and therefore

$$E\left(\hat{\theta}_f|\mu, data\right) = \frac{(m-1)}{m}\frac{n(\bar{x}-\mu)}{(n-1)}.$$

Since

$$p(\mu|data) = \tilde{K}(\bar{x}-\mu)^{-n} \qquad 0<\mu<x_{(1)}$$

it follows that

$$E(\bar{x}-\mu) = \tilde{K}L$$

and

$$E\left(\hat{\theta}_f|\mu, data\right) = \frac{n(m-1)}{m(n-1)}\tilde{K}L.$$

Variance of $\hat{\theta}_f$

$$\begin{aligned} Var\left(\hat{\theta}_f|\mu, data\right) &= E_{\theta|\mu}\left\{Var\left(\hat{\theta}_f|\mu,\theta\right)\right\} + Var_{\theta|\mu}\left[E\left(\hat{\theta}_f|\mu,\theta\right)\right] \\ &= \tfrac{(m-1)}{m^2}\tfrac{\{n(\bar{x}-\mu)\}^2}{(n-1)(n-2)} + \tfrac{(m-1)^2}{m^2}\tfrac{\{n(\bar{x}-\mu)\}^2}{(n-1)^2(n-2)} \\ &= \tfrac{(m-1)n^2}{m^2(n-1)^2(n-2)}(m+n-2)(\bar{x}-\mu)^2. \end{aligned}$$

Further

$$Var\left(\hat{\theta}_f|data\right) = E_\mu\left\{Var\left(\hat{\theta}_f|\mu, data\right)\right\} + Var_\mu\left\{E\left(\hat{\theta}_f|\mu, data\right)\right\}$$

and

$$E_\mu\left\{Var\left(\hat{\theta}_f|data\right)\right\} = \frac{(m-1)n^2}{m^2(n-1)^2(n-2)}(m+n-2)\tilde{K}M.$$

Also

$$Var_\mu\left\{E\left(\hat{\theta}_f|\mu, data\right)\right\} = \frac{(m-1)^2}{m^2}\frac{n^2}{(n-1)^2}Var(\mu|data)$$

and therefore

$$\begin{aligned} Var_\mu\left\{E\left(\hat{\theta}_f|\mu, data\right)\right\} &= \tfrac{(m-1)^2}{m^2}\tfrac{n^2}{(n-1)^2}E\left\{\mu-\left(\bar{x}-\tilde{K}L\right)\right\}^2 \\ &= \tfrac{(m-1)^2n^2}{m^2(n-1)^2}E\left\{(\bar{x}-\mu)^2 - 2\tilde{K}L(\bar{x}-\mu)+\tilde{K}^2L^2\right\} \\ &= \tfrac{(m-1)^2n^2}{m^2(n-1)^2}\left\{\tilde{K}M - 2\tilde{K}L\tilde{K}L + \tilde{K}^2L^2\right\} \\ &= \tfrac{(m-1)^2n^2}{m^2(n-1)^2}\left\{\tilde{K}M - \tilde{K}^2L^2\right\}. \end{aligned}$$

From this it follows that

$$
\begin{aligned}
Var\left(\hat{\theta}_f|data\right) &= \tfrac{(m-1)n^2}{m^2(n-1)^2(n-2)}\left(m+n-2\right)\tilde{K}M + \tfrac{(m-1)^2n^2}{m^2(n-1)^2}\left\{\tilde{K}M-\tilde{K}^2L^2\right\} \\
&= \tfrac{n^2(m-1)}{m(n-1)}\left\{\tfrac{\tilde{K}M}{(n-2)} - \tfrac{(m-1)}{m(n-1)}\tilde{K}^2L^2\right\}.
\end{aligned}
$$

H Proof of Theorem 9.1

Proof of $E\left(U_f|data\right)$

From Equation 9.1 it follows that

$$f\left(U_f|\mu,\theta,\hat{\theta}_f\right) = \left(\frac{m}{\theta}\right)\exp\left\{-\frac{m}{\theta}\left[U_f-\left(\mu-\tilde{k}_2\hat{\theta}_f\right)\right]\right\} \qquad U_f > \mu-\tilde{k}_2\hat{\theta}_f$$

which means that

$$E\left(U_f|\mu,\theta,\hat{\theta}_f\right) = \frac{\theta}{m}+\mu-\tilde{k}_2\hat{\theta}_f.$$

Since

$$\hat{\theta}_f|\theta,\mu \sim \frac{\chi^2_{2m-2}}{2m}\theta$$

it follows that

$$E\left(U_f|\mu,\theta\right) = \mu+\frac{\theta}{m}\left\{1-\tilde{k}_2\left(m-1\right)\right\} = \mu+\frac{\theta}{m}H.$$

Also since the posterior distribution

$$p\left(\theta|\mu,data\right) = \frac{\left\{n\left(\bar{x}-\mu\right)\right\}^n}{\Gamma\left(n\right)}\left(\frac{1}{\theta}\right)^{n+1}\exp\left\{-\frac{n}{\theta}\left(\bar{x}-\mu\right)\right\} \qquad 0<\theta<\infty$$

it follows that

$$E\left(\theta|\mu,data\right) = \frac{n\left(\bar{x}-\mu\right)}{n-1}$$

and therefore

$$E\left(U_f|\mu,data\right) = a\bar{x}H+\mu\left(1-aH\right)$$

where

$$a = \frac{n}{m\left(n-1\right)}.$$

Further

$$p\left(\mu|data\right) = \left(n-1\right)\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1}-\left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1}\left(\bar{x}-\mu\right)^{-n} \qquad 0<\mu<x_{(1)}$$

and therefore

$$E\left(\mu|data\right) = \bar{x}-\tilde{K}L$$

which means that

$$E\left(U_f|data\right) = \bar{x}+\tilde{K}L\left(aH-1\right).$$

Proof of $Var\left(U_f|data\right)$

From Equation 9.1 it follows that

$$Var\left(U_f|\mu,\theta,\hat{\theta}_f\right)=\left(\frac{\theta}{m}\right)^2.$$

Now

$$\begin{aligned}Var\left(U_f|\mu,\theta\right) &= Var_{\hat{\theta}_f}\left\{E\left(U_f|\mu,\theta,\hat{\theta}_f\right)\right\}+E_{\hat{\theta}_f}\left\{Var\left(U_f|\mu,\theta,\hat{\theta}_f\right)\right\}\\ &= Var_{\hat{\theta}_f}\left(\tfrac{\theta}{m}+\mu-\tilde{k}_2\hat{\theta}_f\right)+E_{\hat{\theta}_f}\left\{\left(\tfrac{\theta}{m}\right)^2\right\}\\ &= \tilde{k}_2^2 Var\left(\tfrac{\chi^2_{2m-2}}{2m}\theta\right)+\left(\tfrac{\theta}{m}\right)^2\\ &= \left(\tfrac{\theta}{m}\right)^2\left\{\tilde{k}_2^2\left(m-1\right)+1\right\}=\left(\tfrac{\theta}{m}\right)^2 J.\end{aligned}$$

Further

$$\begin{aligned}Var\left(U_f|\mu,data\right) &= E_{\theta|\mu}\left\{Var\left(U_f|\mu,\theta\right)\right\}+Var_{\theta|\mu}\left\{E\left(U_f|\mu,\theta\right)\right\}\\ &= E_{\theta|\mu}\left\{\left(\tfrac{\theta}{m}\right)^2 J\right\}+Var_{\theta|\mu}\left\{\mu+\tfrac{\theta}{m}H\right\}\\ &= \tfrac{J}{m^2}E\left(\theta^2|\mu,data\right)+\tfrac{H^2}{m^2}Var\left(\theta|\mu,data\right).\end{aligned}$$

Since

$$E\left(\theta^2|\mu,data\right)=\frac{\left\{n\left(\bar{x}-\mu\right)\right\}^2}{\left(n-1\right)\left(n-2\right)}$$

and

$$Var\left(\theta|\mu,data\right)=\frac{\left\{n\left(\bar{x}-\mu\right)\right\}^2}{\left(n-1\right)^2\left(n-2\right)}$$

it follows that

$$Var\left(U_f|\mu,data\right)=\frac{n^2\left(\bar{x}-\mu\right)^2}{m^2\left(n-1\right)\left(n-2\right)}\left\{J+\frac{H^2}{n-1}\right\}.$$

Finally

$$E_\mu\left\{Var\left(U_f|\mu,data\right)\right\}=\frac{n^2}{m^2\left(n-1\right)\left(n-2\right)}\left\{J+\frac{H^2}{n-1}\right\}E\left(\bar{x}-\mu\right)^2.$$

From $p\left(\mu|data\right)$ it follows that

$$E\left(\bar{x}-\mu\right)^2=\tilde{K}M$$

and

$$E_\mu\left\{Var\left(U_f|\mu,data\right)\right\}=\frac{n^2}{m^2\left(n-1\right)\left(n-2\right)}\left\{J+\frac{H^2}{n-1}\right\}\tilde{K}M.$$

Also

$$\begin{aligned} Var_\mu\left\{E\left(U_f|\mu, data\right)\right\} &= Var\left\{a\bar{x}H + \mu\left(1 - aH\right)\right\} \\ &= \left(1 - aH\right)^2 Var\left(\mu|data\right) \\ &= \left(1 - aH\right)^2 \left(\tilde{K}M - \tilde{K}^2L^2\right). \end{aligned}$$

Therefore

$$Var\left(U_f|data\right) = \frac{n^2}{m^2\left(n-1\right)\left(n-2\right)}\left\{J + \frac{H^2}{n-1}\right\}\tilde{K}M + \left(1 - aH^2\right)\left\{\tilde{K}M - \tilde{K}^2L^2\right\}.$$