Bayesian Control Charts for the Two-parameter Exponential Distribution if the Location Parameter Can Take on Any Value Between Minus Infinity and Plus Infinity

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Abstract

By using data that are the mileages for some military personnel carriers that failed in service given by Grubbs (1971) and Krishnamoorthy and Mathew (2009) a Bayesian procedure is applied to obtain control limits for the location and scale parameters, as well as for a one-sided upper tolerance limit in the case of the two-parameter exponential distribution. A comparison between the assumptions of $-\infty < \mu < \infty$ and $0 < \mu < \infty$ are also made. An advantage of the upper tolerance limit is that it monitors the location and scale parameter at the same time. By using Jeffreys' non-informative prior, the predictive distributions of future maximum likelihood estimators of the location and scale parameters are derived analytically. The predictive distributions are used to determine the distribution of the "run-length" and expected "run-length". This paper illustrates the flexibility and unique features of the Bayesian simulation method.

Keywords: Jeffreys' prior, two-parameter exponential, Bayesian procedure, run-length, control chart

1 Introduction

In this section the same notation will be used as given in Krishnamoorthy and Mathew (2009) with the exception that the location parameter can now take on values between $-\infty$ and ∞, similarly as in some literature, see for example Johnson and Kotz (1970).

Therefore the two-parameter exponential distribution has the probability density function

$$f(x;\mu,\theta) = \frac{1}{\theta}\exp\left\{-\frac{(x-\mu)}{\theta}\right\} \qquad x > \mu, \quad -\infty < \mu < \infty, \ \theta > 0$$

where μ is the location parameter and θ the scale parameter.

As before, let $X_1, X_2, \ldots, X_n$ be a sample of n observations from the two-parameter exponential distribution. The maximum likelihood estimators for μ and θ are given by

$$\hat{\mu} = X_{(1)}$$

and

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - X_{(1)}\right) = \bar{X} - X_{(1)}$$

where $X_{(1)}$ is the minimum or the first order statistic of the sample. It is well known (see Johnson and Kotz (1970); Lawless (1982); Krishnamoorthy and Mathew (2009)) that $\hat{\mu}$ and $\hat{\theta}$ are independently distributed with

$$\frac{(\hat{\mu}-\mu)}{\theta} \sim \frac{\chi_2^2}{2n} \text{ and } \frac{\hat{\theta}}{\theta} \sim \frac{\chi_{2n-2}^2}{2n}. \tag{1.1}$$

Let $\hat{\mu}_0$ and $\hat{\theta}_0$ be observed values of $\hat{\mu}$ and $\hat{\theta}$ then it follows from Equation 1.1 that a generalized pivotal quantity (GPQ) for μ is given by

$$G_\mu = \hat{\mu}_0 - \frac{\chi_2^2}{\chi_{2n-2}^2}\hat{\theta}_0 \tag{1.2}$$

and a GPQ for θ is given by

$$G_\theta = \frac{2n\hat{\theta}_0}{\chi^2_{2n-2}} \tag{1.3}$$

From a Bayesian perspective it will be shown that G_μ and G_θ are actually the posterior distributions of μ and θ if the prior $p(\mu,\theta) \propto \theta^{-1}$ is used.

2 Bayesian Procedure

In this section it will be shown that the Bayesian procedure is the same as the generalized variable approach.

If a sample of n observations are drawn from the two-parameter exponential distribution, then the likelihood function is given by

$$L(\mu,\theta|data) = \left(\frac{1}{\theta}\right)^n \exp\left\{-\frac{1}{\theta}\sum_{i=1}^{n}(x_i-\mu)\right\}.$$

As prior the Jeffreys' prior

$$p(\mu,\theta) \propto \theta^{-1}$$

will be used.

The joint posterior distribution of μ and θ is

$$\begin{aligned} p(\theta,\mu|data) &\propto p(\mu,\theta)\,L(\mu,\theta|data) \\ &= K_1\left(\tfrac{1}{\theta}\right)^{n+1}\exp\left\{-\tfrac{n}{\theta}(\bar{x}-\mu)\right\} \quad -\infty<\mu<x_{(1)},\ 0<\theta<\infty \end{aligned} \tag{2.1}$$

It can easily be shown that

$$K_1 = \frac{n^n\left(\hat{\theta}\right)^{n-1}}{\Gamma(n-1)} \text{ where } \hat{\theta} = \bar{x}-x_{(1)}.$$

The posterior distribution of μ is

$$\begin{aligned} p(\mu|data) &= \textstyle\int_0^\infty p(\theta,\mu|data)\,d\theta \\ &= (n-1)\left(\hat{\theta}\right)^{n-1}\left(\tfrac{1}{\bar{x}-\mu}\right)^n \qquad -\infty<\mu<x_{(1)} \end{aligned} \tag{2.2}$$

and the posterior distribution of θ is

$$\begin{aligned} p(\theta|data) &= \textstyle\int_{-\infty}^{x_{(1)}} p(\theta,\mu|data)\,d\mu \\ &= K_2\left(\tfrac{1}{\theta}\right)^n\exp\left\{-\tfrac{n\hat{\theta}}{\theta}\right\} \text{ where } 0<\theta<\infty \end{aligned} \tag{2.3}$$

an Inverse Gamma distribution where

$$K_2 = \frac{n^{n-1}\left(\hat{\theta}\right)^{n-1}}{\Gamma\left(n-1\right)}.$$

The conditional posterior distribution of θ given μ is given by

$$\begin{aligned} p\left(\theta|\mu, data\right) &= \tfrac{p(\theta,\mu|data)}{p(\mu|data)} \\ &= K_3\left(\tfrac{1}{\theta}\right)^{n+1}\exp\left\{-\tfrac{n}{\theta}\left(\bar{x}-\mu\right)\right\} \text{ where } 0<\theta<\infty \end{aligned} \tag{2.4}$$

an Inverse Gamma distribution where

$$K_3 = \frac{\left\{n\left(\bar{x}-\mu\right)\right\}^n}{\Gamma\left(n\right)}.$$

Also the conditional posterior distribution of μ given θ is

$$\begin{aligned} p\left(\mu|\theta, data\right) &= \tfrac{p(\theta,\mu|data)}{p(\theta|data)} \\ &= \tfrac{n}{\theta}\exp\left\{-\tfrac{n}{\theta}\left(x_{(1)}-\mu\right)\right\} \text{ where } -\infty<\mu<x_{(1)} \end{aligned} \tag{2.5}$$

The following theorem can now easily be proved:

Theorem 2.1. *The distribution of the generalized pivotal quantities G_μ and G_θ defined in Equations 1.2 and 1.3 are exactly the same as the posterior distributions $p\left(\mu|data\right)$ and $p\left(\theta|data\right)$ given in Equations 2.2 and 2.3.*

Proof. The proof is given in Appendix A. □

3 The Predictive Distributions of Future Sample Location and Scale Maximum Likelihood Estimators, $\hat{\mu}_f$ and $\hat{\theta}_f$

Consider a future sample of m observations from the two-parameter exponential population: $X_{1f}, X_{2f}, \ldots, X_{mf}$. The future sample mean is defined as $\bar{X}_f = \frac{1}{m}\sum_{j=1}^{m} X_{jf}$. The smallest value in the sample is denoted by $\hat{\mu}_f$ and $\hat{\theta}_f = \bar{X}_f - \hat{\mu}_f$. To obtain control charts for $\hat{\mu}_f$ and $\hat{\theta}_f$ their predictive distributions must first be derived. If $-\infty<\mu<\infty$, then the descriptive statistics, posterior distributions and predictive distributions will be denoted by a tilde (~).

Theorem 3.1. *The predictive distribution of a future sample location maximum likelihood estimator, $\hat{\mu}_f$ is given by*

$$\tilde{f}\left(\hat{\mu}_f|data\right) = \begin{cases} \tilde{K}^*\left[\frac{1}{n(\bar{x}-\hat{\mu}_f)}\right]^n & -\infty<\hat{\mu}_f<x_{(1)} \\ \tilde{K}^*\left[\frac{1}{n\hat{\theta}+m\left(\hat{\mu}_f-x_{(1)}\right)}\right]^n & x_{(1)}<\hat{\mu}_f<\infty \end{cases} \tag{3.1}$$

where

$$\tilde{K}^* = \frac{n^n\left(n-1\right)m}{n+m}\left(\hat{\theta}\right)^{n-1}.$$

Proof. The proof is given in Appendix B. □

The reason why $\tilde{f}\left(\hat{\mu}_f|data\right)$ (Equation 3.1) differs from $f\left(\hat{\mu}_f|data\right)$ is that it is assumed that $0<\mu<\infty$ which results in a posterior distribution of $p\left(\mu|data\right)=\left(n-1\right)\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1}-\left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1}\left(\bar{x}-\mu\right)^{-n}$, while for Equation 3.1 it is assumed that $-\infty<\mu<\infty$ and the posterior distribution for μ is therefore $\tilde{p}\left(\mu|data\right)\propto\left(n-1\right)\left(\hat{\theta}\right)^{n-1}\left(\bar{x}-\mu\right)^{-n}$.

Theorem 3.2. *The mean and variance fo $\hat{\mu}_f$ is given by*

$$\tilde{E}\left(\hat{\mu}_f|data\right)=\bar{x}-\frac{mn-m-n}{m\left(n-2\right)}\hat{\theta} \tag{3.2}$$

and

$$\tilde{Var}\left(\hat{\mu}_f|data\right)=\frac{n^3\left(n-2\right)+\left(mn-m-n\right)^2}{m^2\left(n-1\right)\left(n-3\right)\left(n-2\right)^2}\left(\hat{\theta}\right)^2 \tag{3.3}$$

Proof. By deleting the term $\left(\frac{1}{\bar{x}}\right)$ from results from the previous report where $0<\mu<\infty$ Equation 3.2 and Equation 3.3 follows. □

Theorem 3.3. *The predictive distribution of a future sample scale maximum likelihood estimator, $\hat{\theta}_f$ is given by*

$$\tilde{f}\left(\hat{\theta}_f|data\right)=\frac{\Gamma\left(m+n-2\right)m^{m-1}\left(n\hat{\theta}\right)^{n-1}\left(\hat{\theta}_f\right)^{m-2}}{\Gamma\left(m-1\right)\Gamma\left(n-1\right)\left(m\hat{\theta}_f+n\hat{\theta}\right)^{m+n-2}}\qquad 0<\hat{\theta}_f<\infty \tag{3.4}$$

Proof. The proof is given in Appendix C. □

Corollary 3.4. $\hat{\theta}_f|data\sim\hat{\theta}\frac{n}{m}\frac{(m-1)}{(n-1)}F_{2m-2,2n-2}$.

Proof. The proof is given in Appendix D. □

Theorem 3.5. *The mean and variance of $\hat{\theta}_f$ is given by*

$$\tilde{E}\left(\hat{\theta}_f|data\right)=\frac{\left(m-1\right)}{m}\frac{n}{\left(n-2\right)}\hat{\theta} \tag{3.5}$$

and

$$\tilde{var}\left(\hat{\theta}_f|data\right)=\frac{n^2\left(m-1\right)}{m^2\left(n-2\right)^2\left(n-3\right)}\left(n+m-3\right)\hat{\theta}^2 \tag{3.6}$$

Proof. By deleting the term $\left(\frac{1}{\bar{x}}\right)$ from results from the previous report where $0<\mu<\infty$, Equation 3.5 and Equation 3.6 follow. □

4 Example

The following data is given in Grubbs (1971) as well as in Krishnamoorthy and Mathew (2009). The failure mileages given in Table 4.1 fit a two-parameter exponential distribution.

Table 4.1: Failure Mileages of 19 Military Carriers

162	200	271	302	393	508	539	629	706	777
884	1008	1101	1182	1463	1603	1984	2355	2880	

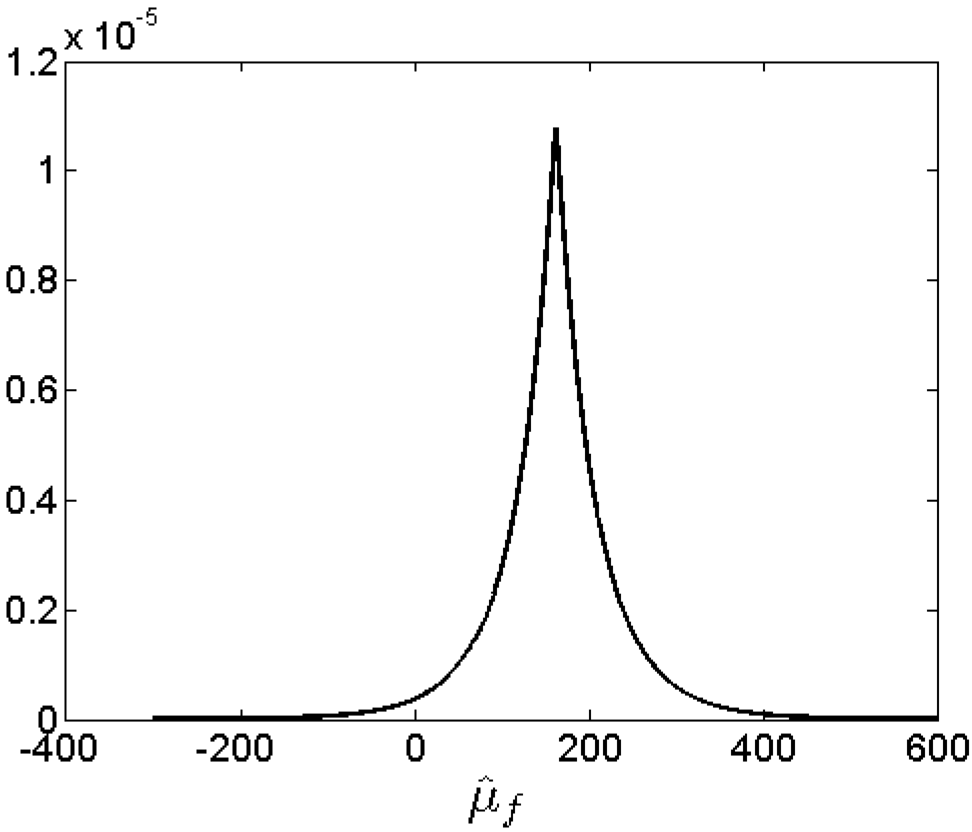
For this data, the maximum likelihood estimates are $\hat{\mu} = x_{(1)} = 162$, $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - x_{(1)}\right) = \bar{x} - x_{(1)} = 835.21$ and $n = 19$.

As mentioned in the introductory section, the aim of this article is to obtain control charts for location and scale maximum likelihood estimates as well as for a one-sided upper tolerance limit.

4.1 The Predictive Distribution of $\hat{\mu}_f$ $(-\infty < \mu < \infty)$

By using Equation 3.1 the predictive distribution $\tilde{f}\left(\hat{\mu}_f | data\right)$ for $m = 19$ future failure mileage data is illustrated in Figure 4.1.

Figure 4.1: Distribution of $\hat{\mu}_f$, $n = 19$, $m = 19$



$$\tilde{mean}\left(\hat{\mu}_f\right) = \tilde{median}\left(\hat{\mu}_f\right) = \tilde{mode}\left(\hat{\mu}_f\right) = 162,\ \tilde{var}\left(\hat{\mu}_f\right) = 5129.2$$
$$95\%\ \tilde{interval}\left(\hat{\mu}_f\right) = (11.2; 312.8)$$
$$99.73\%\ \tilde{interval}\left(\hat{\mu}_f\right) = (-154.85; 477.36)$$
$$96.08\%\ \tilde{interval}\left(\hat{\mu}_f\right) = (-2; 326)$$

For $-\infty < \mu < \infty$, $n = 19$, $m = 19$, the predictive distribution $\tilde{p}(\hat{\mu}_f|data)$ is symmetrical. $\tilde{mean}(\hat{\mu}_f) = \tilde{median}(\hat{\mu}_f) = \tilde{mode}(\hat{\mu}_f) = 162$. $\tilde{Mode}(\hat{\mu}_f)$ for $-\infty < \mu < \infty$ is exactly the same as that for $0 < \mu < \infty$. A further comparison of Figure 4.1 and results from previous report shows that $\tilde{var}(\hat{\mu}_f) = 5129.2$ is somewhat larger that $var(\hat{\mu}_f) = 3888.7$, when $0 < \mu < \infty$. Also the predictive intervals are somewhat wider. A dissatisfactory aspect for $-\infty < \mu < \infty$ is that the 99.73% $\tilde{interval}(\hat{\mu}_f) = (-154.85; 477.36)$, i.e., contains negative values.

In Table 4.2 descriptive statistics are given for the run-length and expected run-length.

Table 4.2: Descriptive Statistics for the Run-length and Expected Run-length in the case of $\hat{\mu}_f$; $-\infty < \mu < \infty$ and $\beta = 0.039$ for $n = 19$ and $m = 19$

Descriptive Statistics	$f(r\|data)$ Equal Tail	Expected Run-length Equal Tail
$\tilde{mean}$	367.84	373.34
$\tilde{median}$	44.50	69.93
$\tilde{var}$	6.663×10^7	1.414×10^8
95% $\tilde{interval}$	$(0; 740.50)$	$(0; 724.64)$

A comparison of Table 4.2 with results from previous report shows that if $-\infty < \mu < \infty$ the expected (mean) run-length ≈ 370 if $\beta = 0.0392$ while this is the case if $\beta = 0.0258$ for $0 < \mu < \infty$. Also the 95% intervals are somewhat shorter for $-\infty < \mu < \infty$.

4.2 A Comparison of the Predictive Distributions for $\hat{\theta}_f$

In Figure 4.2 comparisons are made between

$$\tilde{f}\left(\hat{\theta}_f|data\right) = \frac{\Gamma(m+n-2)\, m^{m-1}\left(n\hat{\theta}\right)^{n-1}\left(\hat{\theta}_f\right)^{m-2}}{\Gamma(m-1)\,\Gamma(n-1)\left(m\hat{\theta}_f + n\hat{\theta}\right)^{m+n-2}} \qquad 0 < \hat{\theta}_f < \infty$$

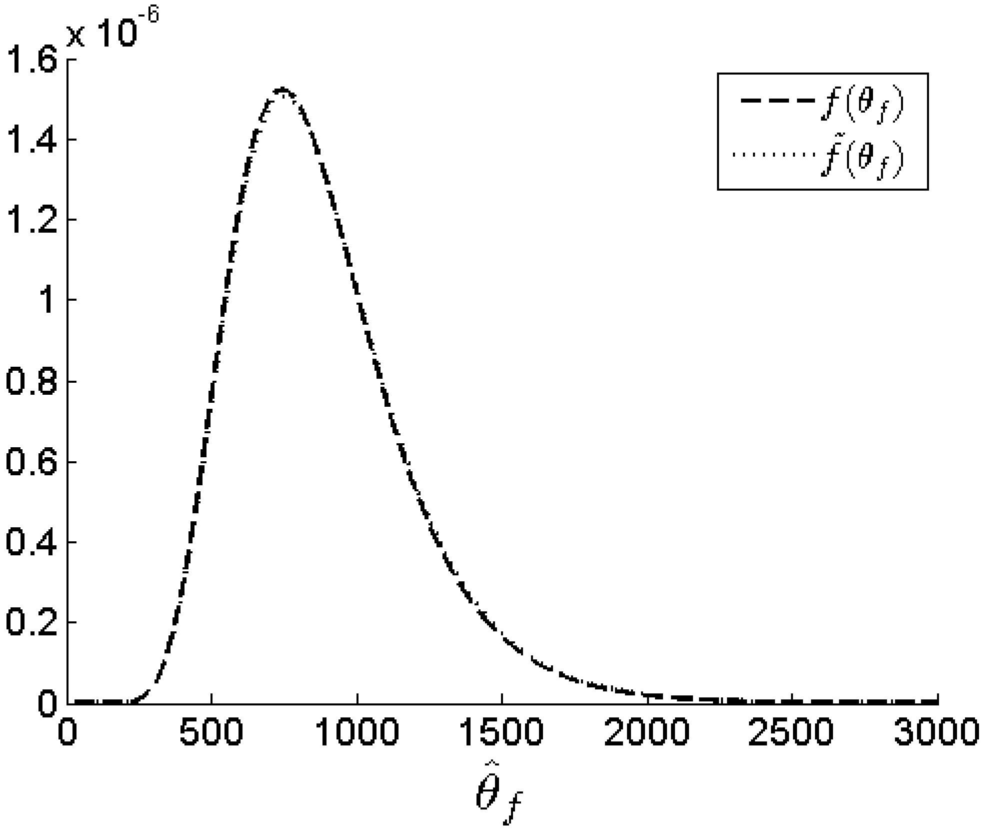
and

$$f\left(\hat{\theta}_f|data\right) = m^{m-1}n^{n-1}\frac{\Gamma(m+n-2)}{\Gamma(m-1)\,\Gamma(n-1)}\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1} - \left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1}\left(\hat{\theta}_f\right)^{m-2}$$

$$\times\left\{\left(\frac{1}{m\hat{\theta}_f + n\hat{\theta}}\right)^{m+n-2} - \left(\frac{1}{m\hat{\theta}_f + n\bar{x}}\right)^{m+n-2}\right\} \qquad 0 < \hat{\theta}_f < \infty.$$

As mentioned $\tilde{f}\left(\hat{\theta}_f|data\right)$ denotes the predictive density function if $-\infty < \mu < \infty$ and $f\left(\hat{\theta}_f|data\right)$ denotes the predictive density function if $0 < \mu < \infty$.

Figure 4.2: $f\left(\hat{\theta}_f|data\right)$, $n=19$, $m=19$



In Table 4.3 descriptive statistics are given for $\tilde{f}\left(\hat{\theta}_f|data\right)$ and $f\left(\hat{\theta}_f\right)$.

Table 4.3: Descriptive Statistics of $\tilde{f}\left(\hat{\theta}_f|data\right)$ and $f\left(\hat{\theta}_f|data\right)$

Descriptive Statistics	$\tilde{f}\left(\hat{\theta}_f\|data\right)$	$f\left(\hat{\theta}_f\|data\right)$
$Mean\left(\hat{\theta}_f\right)$	884.34	876.98
$Median\left(\hat{\theta}_f\right)$	835.2	829.1
$Mode\left(\hat{\theta}_f\right)$	747.3	743.3
$Var\left(\hat{\theta}_f\right)$	95037	91991
95% $Equal-tail\ Interval\left(\hat{\theta}_f\right)$	$(430;1622)$ Length=1192	$(428;1601)$ Length=1173
95% $HPD\ Interval\left(\hat{\theta}_f\right)$	$(370.8;1500.5)$ Length=1129.7	$(369.2;1482.6)$ Length=1113.4

Als, the exact means and variances are given by:

$$\tilde{E}\left(\hat{\theta}_f|data\right)=\frac{n(m-1)}{m(n-2)}\hat{\theta}=884.34,$$

$$\tilde{Var}\left(\hat{\theta}_f\right)=\frac{n^2(m-1)}{m^2(n-2)^2(n-3)}(n+m-3)\,\hat{\theta}^2=95042,$$

$$E\left(\hat{\theta}_f|data\right) = \frac{n\,(m-1)}{m\,(n-2)}\tilde{K}L = 976.98,$$

$$Var\left(\hat{\theta}_f|data\right) = \frac{n^2\,(m-1)}{m\,(n-1)}\left\{\frac{\tilde{K}M}{n-2} - \frac{m-1}{m\,(n-1)}\tilde{K}^2L^2\right\} = 91991$$

where

$$\tilde{K} = (n-1)\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1} - \left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1},$$

$$L = \frac{1}{n-2}\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-2} - \left(\frac{1}{\bar{x}}\right)^{n-2}\right\}$$

and

$$M = \frac{1}{n-3}\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-3} - \left(\frac{1}{\bar{x}}\right)^{n-3}\right\}.$$

From Figure 4.2 and Table 4.3 it can be seen that $\tilde{f}\left(\hat{\theta}_f|data\right)$ and $f\left(\hat{\theta}_f|data\right)$ are for all practical purposes the same. Also the exact means and variances are very much the same as the numerical values. It therefore seems that whether the assumption is that $-\infty < \mu < \infty$ or that $0 < \mu < \infty$ does not play a big role in the prediction of $\hat{\theta}_f$.

In Table 4.4 comparisons are made between the run-lengths and expected run-lengths in the case of $\hat{\theta}_f$ for $-\infty < \mu < \infty$ and $0 < \mu < \infty$.

Table 4.4: Descriptive Statistics for the Run-length and Expected Run-lengths in the case of $\hat{\theta}_f$, $-\infty < \mu < \infty$, $0 < \mu < \infty$ and $\beta = 0.018$

Descriptive Statistics	$-\infty < \mu < \infty$		$0 < \mu < \infty$	
	$f\,(r\|data)$ Equal Tail	Expected Run-Length Equal Tail	$f\,(r\|data)$ Equal Tail	Expected Run-Length Equal Tail
Mean	369.29	370.42	375.25	375.25
Median	122.8	229.28	127.4	238.38
Variance	3.884×10^5	1.2572×10^5	3.9515×10^5	1.2698×10^5
95% Interval	$(0; 1586)$	$(7.423; 1089.7)$	$(0; 1602.5)$	$(7.809; 1096.7)$

From Table 4.4 it can be seen that the corresponding statistics for $-\infty < \mu < \infty$ and $0 < \mu < \infty$ are for all practical purposes the same. So with respect to $\hat{\theta}_f$ it does not really matter whether it is assumed that $\mu$is positive or not.

5 Phase I Control Chart for the Scale Parameter in the Case of the Two-parameter Exponential Distribution

Statistical quality control is implemented in two phases. In Phase I the primary interest is to assess process statbility. Phase I is the so-called retrospective phase and Phase II the prospective or monitoring phase. The construction of Phase I control charts should be considered as a multiple testing problem. The distribution of a set of dependent variables (ratios of chi-square random variables) will therefore be used to calculate the control limits so that the false alarm probability (FAP) is not larger than $FAP_0 = 0.05$. To obtain control limits in Phase I, more than one sample is needed. Therefore in the example that follows there will be $m = 5$ samples for a subgroup each of size $n = 10$.

Example 5.1

The data in Table 5.1 are simulated data obtaind from the following two-parameter exponential distribution:

$$f(x_{ij};\theta,\mu) = \frac{1}{\theta}\exp\left\{-\frac{x_{ij}-\mu_i}{\theta}\right\},\ i=1,2,\ldots,m,\ j=1,2,\ldots,n;\ x_{ij}>\mu_i$$

$$\theta = 8;\ \mu_i = 2i;\ m = 5;\ n = 10.$$

Table 5.1: Simulated Data for the Two-parameter Exponential Distribution

μ_i				
2	**4**	**6**	**8**	**10**
3.6393	18.7809	9.3759	10.7846	16.5907
2.7916	4.2388	32.6582	35.5781	17.7079
18.5094	4.3502	7.3084	18.2721	12.1376
2.749	9.7827	6.5463	32.6032	11.8333
5.6664	5.7823	9.1002	26.6535	23.4186
20.6199	19.6218	8.2193	9.5539	15.7106
12.2267	10.9065	8.3750	10.9127	16.4669
6.8282	4.7042	13.4873	17.1883	13.4918
2.3474	5.8636	9.3791	8.4085	12.7471
2.2859	4.3308	20.1200	34.9469	12.2616

From the simulated data in Table 5.1 we have

$$\hat{\theta}_{i\cdot} = \bar{X}_{i\cdot} - X_{(1,i)} = \begin{bmatrix} 5.4780 & 4.5974 & 5.9107 & 12.0817 & 34.023 \end{bmatrix},$$

$$\sum_{i=1}^{m}\hat{\theta}_{i\cdot} = 31.4701$$

and

$$\hat{\theta} = \frac{1}{m}\sum_{i=1}^{m}\hat{\theta}_{i\cdot} = 6.2940.$$

It is well known that $\hat{\theta}_i \sim \frac{\theta}{2n}\chi^2_{2n-2} = \frac{\theta}{2n}Y_i$ and therefore $\sum_{i=1}^{m}\hat{\theta}_i \sim \frac{\theta}{2n}\sum_{i=1}^{m}Y_i$.

Let

$$Z_1 = \frac{\hat{\theta}_i}{\sum_{i=1}^{m}\hat{\theta}_i} = \frac{\frac{\theta}{2n}Y_i}{\frac{\theta}{2n}\sum_{i=1}^{m}Y_i} = \frac{Y_i}{\sum_{i=1}^{m}Y_i} \quad i = 1, 2, \ldots, m$$

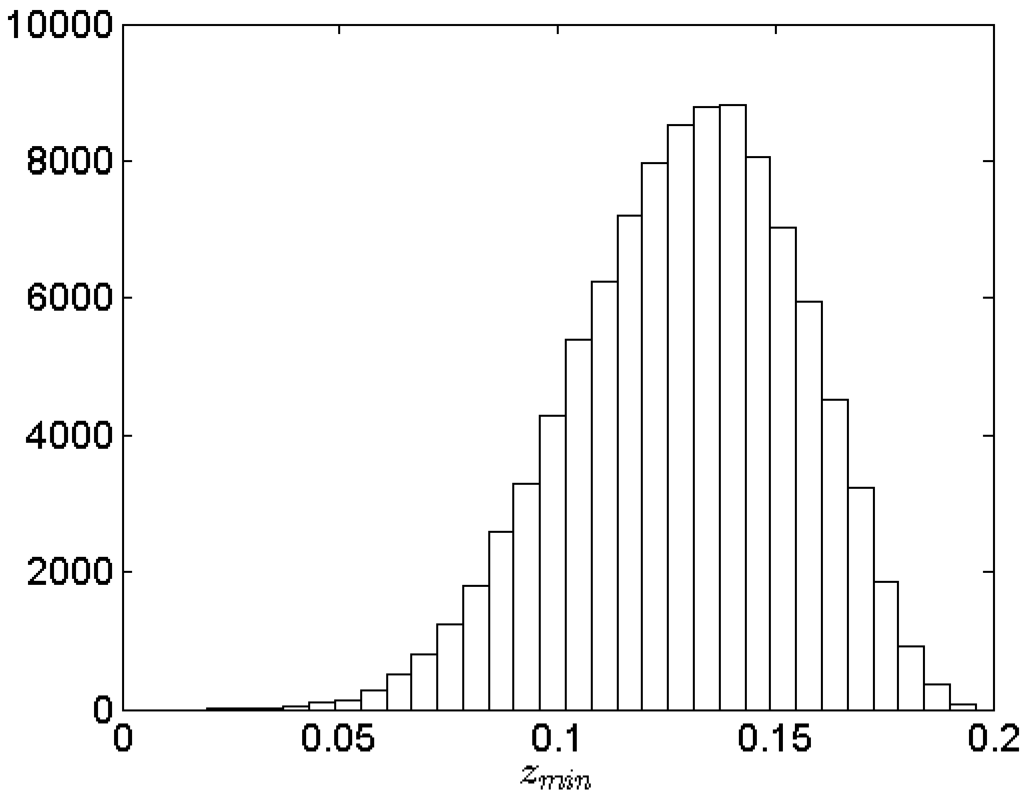
where

$$Y_i \sim \chi^2_{2n-2}.$$

For further details see also Human, Chakraborti, and Smit (2010).

To obtain a lower control limit for the data in Table 5.1, the distribution of $Z_{min} = min\,(Z_1, Z_2, \ldots, Z_m)$ must be obtained.

Figure 5.1: Distribution of $Z_{min} = min\,(Z_1, Z_2, \ldots, Z_m)$, 100 000 simulations



The distribution of Z_{min} obtained from 100 000 simulations is illustrated in Figure 5.1. The value $Z_{0.05} = 0.0844$ is calculated such that the FAP is at a level of 0.05. The lower control limit is then determined as

$$LCL = Z_{0.05}\sum_{i=1}^{m}\hat{\theta}_i = (0.0844)(31.4701) = 2.656.$$

Since $\hat{\theta}_i > 2.656 \;\; (i = 1, 2, \ldots, m)$ it can be concluded that the scale parameter is under statistical control.

6 Lower Control Limit for the Scale Parameter in Phase II

In the first part of this section, the lower control limit in a Phase II setting will be derived using the Bayesian predictive distribution.

The following theorem can easily be proved:

Theorem 6.1. *For the two-parameter exponential distribution*

$$f\left(x_{ij};\theta,\mu_i\right)=\left(\frac{1}{\theta}\right)\exp\left\{-\frac{1}{\theta}\left(x_{ij}-\mu_i\right)\right\}\quad i=1,2,\ldots,m,\ j=1,2,\ldots,n,\ \textit{and } x_{ij}>\mu_i$$

the posterior distribution of the parameter θ given the data is given by

$$p\left(\theta|data\right)=\frac{\left(n\hat{\theta}\right)^{m(n-1)}}{\Gamma\left(m\left(n-1\right)\right)}\left(\frac{1}{\theta}\right)^{m(n-1)+1}\exp\left(-\frac{n\hat{\theta}}{\theta}\right)\qquad\theta>0$$

an Inverse Gamma Distribution.

Proof. The proof is given in Appendix E. □

Theorem 6.2. *Let $\hat{\theta}_f$ be the maximum likelihood estimator of the scale parameter in a future sample of n observations, then the predictive distribution fo $\hat{\theta}_f$ is*

$$f\left(\hat{\theta}_f|data\right)=\frac{\Gamma\left[m\left(n-1\right)+n-1\right]}{\Gamma\left(n-1\right)\Gamma\left[m\left(n-1\right)\right]}\frac{\left(\hat{\theta}_f\right)^{n-2}}{\left(\hat{\theta}_f+\hat{\theta}\right)^{m(n-1)+n-1}}\qquad\hat{\theta}_f>0$$

which means that

$$\hat{\theta}_f|data\sim\frac{\hat{\theta}}{m}F_{2(n-1);2m(n-1)}$$

where

$$\hat{\theta}=\sum_{i=1}^{m}\hat{\theta}_i.$$

Proof. The proof is given in Appendix F. □

At $\beta=0.0027$ the lower control limit is obtained as $\frac{\hat{\theta}}{m}F_{2(n-1);2m(n-1)}\left(0.0027\right)=\frac{31.4701}{5}\left(0.29945\right)=1.8847$ for $m=5$ and $n=10$.

Assuming that the process remains stable, the predictive distribution for $\hat{\theta}_f$ can als be used to derive the distribution of the run-length, that is the number of samples until the control chart signals for the first time.

The resulting region of size β using the predictive distribution for the determination of the run-lenght is defined as

$$\beta=\int_{R(\beta)}f\left(\hat{\theta}_f|data\right)d\hat{\theta}_f$$

where

$$R(\beta) = (0; 1.8847)$$

is the lower one-sided control interval.

Given θ and a stable process, the distribution of the run-length r is Geometric with parameter

$$\psi(\theta) = \int_{R(\beta)} f\left(\hat{\theta}_f|\theta\right) d\hat{\theta}_f$$

where $f\left(\hat{\theta}|\theta\right)$ is the distribution of a future sample scale parameter estimator given θ.

The value of the parameter θ is however unknown and its uncertainty is described by the posterior distribution $p(\theta|data)$.

The following theorem can also be proved.

Theorem 6.3. *For given θ the parameter of the Geometric distribution is*

$$\psi(\theta) = \psi\left(\chi^2_{2m(n-1)}\right)$$

for given $\chi^2_{2m(n-1)}$ which means that the parameter is only dependent on $\chi^2_{2m(n-1)}$ and not on θ.

Proof. The proof is given in Appendix G. □

As mentioned, by simulating θ from $p(\theta|data)$ the probability density function of $f\left(\hat{\theta}_f|\theta\right)$ as well as the parameter $\psi(\theta)$ can be obtained. This must be done for each future sample. Therefore, by simulating a large number of θ values from the posterior distribution a large number of $\psi(\theta)$ values can be obtained.

A large number of Geometric and run-length distributions with different parameter values $(\psi(\theta_1), \psi(\theta_2), \ldots, \psi(\theta_l))$ will therefore be available. The unconditional run-length distribution is obtained by using the Rao-Blackweel method, i.e., the average of the conditional run-length distributions.

In Table 6.1 results for the run-length at $\beta = 0.0027$ for $n = 10$ and different values for m are presented for the lower control limit of the scale parameter estimator.

Table 6.1: Two Parameter Exponential Run Lengths Results

n	m	mean (ARL)	median (ARL)	Mean (PDF)	Median (PDF)	One-sided		Two-sided	
						Low	High	Low	High
10	5	1040.878616	561.9041434	1016.107825	365	120.668063	3285.604745	90.53140623	4875.25148
10	6	859.2946037	528.043143	850.6496099	340	132.1636771	2633.779211	103.814474	3585.574629
10	7	752.7840665	502.7053642	747.964747	330	137.1365507	2163.362738	107.5044166	2913.68755
10	8	692.3117213	478.8131128	688.5627995	320	147.7899272	1904.506951	121.7607631	2547.589774
10	9	644.1233737	467.3792855	640.6357323	310	149.1918895	1707.552491	119.5874038	2198.554651
10	10	616.4008914	456.271994	613.0915372	305	154.9653041	1581.526121	128.5788067	2094.950512
10	11	582.5441167	445.4805119	579.3980237	295	159.4756421	1466.351811	133.3858464	1874.789144
10	12	563.8471571	445.4805119	560.8517002	295	165.742153	1340.987775	139.7090749	1681.424117
10	13	542.293566	440.1999559	539.3968128	290	170.6404787	1264.446813	145.0341248	1534.216048
10	14	523.2584573	429.8629367	520.4260911	285	170.6404787	1193.031076	145.0341248	1466.351811
10	15	514.4424077	429.8629367	511.6810866	285	177.4499555	1142.597024	149.1918895	1381.311827
10	16	498.7108368	424.804032	496.0401877	285	179.2039115	1064.049573	150.6101839	1283.085571
10	17	493.4525335	419.816607	490.7979663	280	179.2039115	1064.049573	154.9653041	1264.446813
10	18	491.9807528	424.804032	489.3787352	285	188.2980341	1019.999561	164.1476967	1228.123456
10	19	479.4053832	414.8994965	476.8198625	280	184.5942927	978.1005367	159.4756421	1175.931545
10	20	470.1638992	410.0515567	467.6270667	275	188.2980341	951.3034035	164.1476967	1110.378304
10	50	408.6804318	386.8108266	406.4862335	265	235.6714776	654.3843849	214.7137467	726.2252763
10	100	389.0488816	377.9609703	386.9577637	260	267.625879	548.0561023	248.3896595	583.4710484
10	500	374.3715597	373.6276458	372.3573508	255	318.8260483	440.1999559	308.3784626	450.8374316
10	1000	372.9795051	373.6276458	370.9756458	255	333.4301277	414.8994965	326.0292806	424.804032
10	5000	371.9360097	373.6276458	369.9394557	255	352.8387861	391.3293518	348.8501358	395.9116345
10	10000	371.7307984	373.6276458	369.7352037	255	356.8826129	386.8108266	356.8826129	386.8108266

mean(ARL) and median(ARL) refer to results obtained from the expected run-length while mean(PDF) and median(PDF) refer to results obtained from the probability density function of the run-length.

From Table 6.1 it can be seen that as the number of samples increase (larger m) the mean and median run-lengths converge to the expected run-length of 370.

Further define $\bar{\psi}(\theta) = \frac{1}{l}\sum_{i=1}^{l}\psi(\theta_i)$. From Menzefricke (2002) it is known that as $l \to \infty$, $\bar{\psi}(\theta) \to \beta = 0.0027$ and the harmonic mean of the unconditional run-length will be $\left(\frac{1}{\beta}\right) = \frac{1}{0.0027} = 370$. Therefore it does not matter how small m and n is, the harmonic mean of the run-length will always be $\frac{1}{\beta}$ if $l \to \infty$. In the case of the simulated example the mean run-length is 1040.88 and the median run-length 561.90. The reason for these large values is the uncertainty in the parameter estimate because of the small sample size and number of samples ($n = 10$ and $m = 5$). β however can easily be adjusted to get a mean run-length of 370.

7 A Comparison of the Predictive Distributions and Control Charts for a One-sided Upper Tolerance Limit, U_f

A future sample tolerance limit is defined as

$$U_f = \hat{\mu}_f - \tilde{k}_2\hat{\theta}_f \text{ where } \hat{\mu}_f > \mu \text{ and } \hat{\theta}_f > 0.$$

Also

$$f\left(\hat{\mu}_f|\mu,\theta\right) = \left(\frac{m}{\theta}\right)\exp\left\{-\frac{m}{\theta}\left(\hat{\mu}_f - \mu\right)\right\} \;\; \hat{\mu}_f > \mu$$

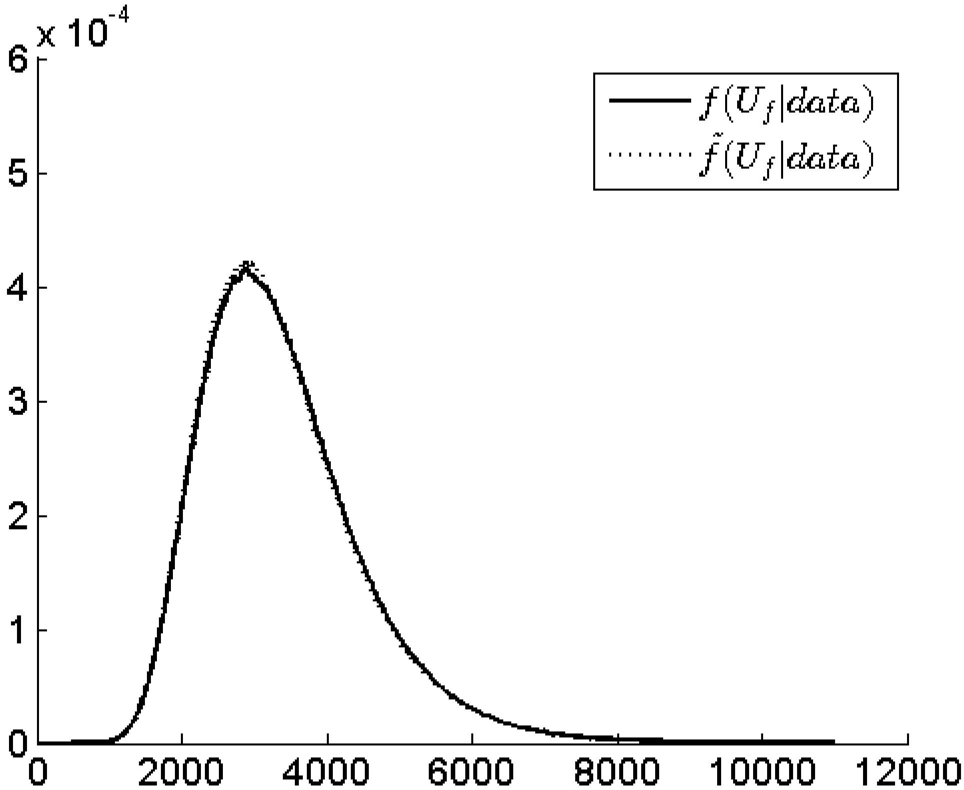
which means that

$$f\left(U_f|\mu,\theta,\hat{\theta}_f\right) = \left(\frac{m}{\theta}\right)\exp\left\{-\frac{m}{\theta}\left[U_f - \left(\mu - \tilde{k}_2\hat{\theta}_f\right)\right]\right\} \qquad U_f > \mu - \tilde{k}_2\hat{\theta}_f \tag{7.1}$$

A comparison will be made between $f\left(U_f|data\right)$ and $\tilde{f}\left(U_f|data\right)$. The difference in the simulation procedure for these two density functions is that in the case of $f\left(U_f|data\right)$ it is assumed that $0 < \mu < \infty$ which results in a posterior distribution of $p\left(\mu|data\right) = (n-1)\left\{\left(\frac{1}{\hat{\theta}}\right)^{n-1} - \left(\frac{1}{\bar{x}}\right)^{n-1}\right\}^{-1}\left(\bar{x}-\mu\right)^{-n}$ while for $\tilde{f}\left(U_f|data\right)$ it is assumed that $-\infty < \mu < \infty$ and the posterior distribution for μ is then $\tilde{p}\left(\mu|data\right) = (n-1)\left(\hat{\theta}\right)^{n-1}\left(\bar{x}-\mu\right)^{-n}$.

In the following figure comparisons are made between $f\left(U_f|data\right)$ and $\tilde{f}\left(U_f|data\right)$.

Figure 7.1: Predictive Densities of $f\left(U_f|data\right)$ and $\tilde{f}\left(U_f|data\right)$



The descriptive statistics obtained from Figure 7.1 are presented in Table 7.1.

Table 7.1: Descriptive Statistics of and $f\left(U_f|data\right)$ and $\tilde{f}\left(U_f|data\right)$

Descriptive Statistics	$f\left(U_f\|data\right)$	$\tilde{f}\left(U_f\|data\right)$
$Mean\left(U_f\right)$	3394.7	3406.6
$Median\left(U_f\right)$	3211.5	3225.8
$Mode\left(\hat{U}_f\right)$	2900	2907
$Var\left(\hat{U}_f\right)$	1.2317×10^6	1.2694×10^6
95% Equal-tail Interval	$(1736.5; 6027)$	$(1738; 6106)$
99.73% Equal-tail Interval	$(1249.05; 7973)$	$(1249.9; 8620)$

Also the exact means and variances for $f\left(U_f|data\right)$ are

$$E\left(U_f|data\right) = \bar{x} + \tilde{K}L\left(aH - 1\right) = 3394.8$$

and

$$Var\left(U_f|data\right) = \frac{n^2}{m^2\left(n-1\right)\left(n-2\right)}\left\{J + \frac{H^2}{n-1}\right\}\tilde{K}M + \left(1 - aH\right)^2\left\{\tilde{K}M - \tilde{K}^2L^2\right\} = 1.2439 \times 10^6$$

where

$$a = \frac{n}{m(n-1)},$$

$$H = 1 - \tilde{k}_2 (m-1),$$

$$J = 1 + \tilde{k}_2^2 (m-1)$$

and $\tilde{K}$, L and M defined as before.

The exact means and variances for $\tilde{f}(U_f|data)$ are

$$\tilde{E}(U_f|data) = \bar{x} + \frac{n-1}{n-2}(aH-1)\hat{\theta} = 3415.0$$

and

$$\tilde{Var}(U_f|data) = \frac{1}{(n-2)(n-3)}\left\{\frac{n^2}{m^2}\left(J + \frac{H^2}{n-1}\right) + (1-aH)^2 \frac{n-1}{n-2}\right\}\left(\hat{\theta}\right)^2 = 1.2911 \times 10^6.$$

$\tilde{E}(U_f|data)$ and $\tilde{Var}(U_f|data)$ are derived from $E(U_f|data)$ and $Var(U_f|data)$ by deleting the term $\left(\frac{1}{\bar{x}}\right)$ in $\tilde{K}$, L and M.

It seems that the predictive intervals for $\tilde{f}(U_f|data)$ are somewhat wider than in the case of $f(U_f|data)$.

In Table 7.2 comparisons are made between the run-lengths and expected run-lengths in the case of U_f for $-\infty < \mu < \infty$ and $0 < \mu < \infty$.

Table 7.2: Descriptive Statistics for the Run-lengths and Expected Run-lengths in the Case of U_f; $-\infty < \mu < \infty$; $0 < \mu < \infty$and $\beta = 0.018$

Descriptive Statistics	$-\infty < \mu < \infty$		$0 < \mu < \infty$	
	$f(r\|data)$ Equal Tail	Expected Run-Length Equal Tail	$f(r\|data)$ Equal Tail	Expected Run-Length Equal Tail
Mean	444.95	444.95	418.68	419.68
Median	136.5	258.12	132.1	248.03
Variance	7.6243×10^5	2.8273×10^5	5.8236×10^5	2.0351×10^5
95% Interval	$(0; 1892.4)$		$(0; 1803.6)$	

It is clear from Table 10.1 that the corresponding statistics do not differ much. The mean, median and variance and 95% interval are however somewhat larger for $-\infty < \mu < \infty$.

8 Conclusion

This paper develops a Bayesian control chart for monitoring the scale parameter, location parameter and upper tolerance limit of a two-parameter exponential distribution. In the Bayesian approach prior knowledge about the unknown parameters is formally incorporated into the process of inference by assigning a prior distribution to the parameters. The information contained in the prior is combined with the likelihood function to obtain the posterior distribution. By using the posterior distribution the predictive distributions of $\hat{\mu}_f$, $\hat{\theta}_f$ and U_f can be obtained.

The theory and results described in this paper have been applied to the failure mileages for military carriers analyzed by Grubbs (1971) and Krishnamoorthy and Mathew (2009). The example illustrates the flexibility and unique features of the Bayesian simulation method for obtaining posterior distributions and "run-lengths" for $\hat{\mu}_f$, $\hat{\theta}_f$ and U_f.

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Mathematical Appendices

A Proof of Theorem 2.1

(a) The posterior distribution $p(\theta|data)$ is the same as the distribution of the pivotal quantity $G_\theta = \frac{2n\hat{\theta}}{\chi^2_{2n-2}}$.

Proof:

Let $Z \sim \chi^2_{2n-2}$

$\therefore f(z) = \frac{1}{2^{n-2}\Gamma(n-1)} z^{n-2} \exp\left\{-\frac{1}{2}z\right\}$

We are interested in the distribution of $\theta = \frac{2n\hat{\theta}}{Z}$.

Therefore $Z = \frac{2n\hat{\theta}}{\theta}$ and $\left|\frac{dZ}{d\theta}\right| = \frac{2n\hat{\theta}}{Z}$.

From this it follows that

$$\begin{aligned} f(G_\theta) &= f(\theta) = \frac{1}{2^{n-1}\Gamma(n-1)} \left(\frac{2n\hat{\theta}}{\theta}\right)^{n-2} \frac{2n\hat{\theta}}{\theta^2} \exp\left\{-\frac{n\hat{\theta}}{\theta}\right\} \\ &= \frac{n^{n-1}(\hat{\theta})^{n-1}}{\Gamma(n-1)} \left(\frac{1}{\theta}\right)^n \exp\left\{-\frac{n\hat{\theta}}{\theta}\right\} = p(\theta|data) \end{aligned}$$

See Equation 2.3.

(b) The posterior distribution $p(\mu|data)$ is the same as the distribution of the pivotal quantity $G_\mu = \hat{\mu} - \frac{\chi^2_2}{\chi^2_{2n-2}}\hat{\theta}$.

Proof:

Let $F = \frac{\chi^2_2/2}{\chi^2_{2n-2}/(2n-2)} \sim F_{2,2n-2}$

$\therefore g(f) = \left(1 + \frac{1}{n-1}f\right)^{-n}$ where $0 < f < \infty$

We are interested in the distribution of $\mu = \hat{\mu}_0 - \frac{2\hat{\theta}}{2n-2}F$ which means that $F = \frac{(n-1)}{\hat{\theta}}(\hat{\mu} - \mu)$ and $\left|\frac{dF}{d\mu}\right| = \frac{(n-1)}{\hat{\theta}}$.

Therefore

$$\begin{aligned} g(\mu) &= \left\{1 + \frac{1}{\hat{\theta}}(\hat{\mu}_0 - \mu)\right\}^{-n} \frac{n-1}{\hat{\theta}} \\ &= (n-1)\,\hat{\theta}^{n-1}\left(\frac{1}{\bar{x}-\mu}\right)^n \text{ where } -\infty < \mu < \hat{\mu} \\ &= p(\mu|data) \end{aligned}$$

See Equation 2.2.

(c) The posterior distribution of p$(\mu|\theta, data)$ is the same as the distribution of the pivotal quantity $G_{\mu|\theta} = \hat{\mu} - \frac{\chi^2_2}{2n}\theta$ (see Equation 1.1).

Proof:

Let $\tilde{Z} \sim \chi_2^2$ then

$g(\tilde{z}) = \frac{1}{2}\exp\left\{-\frac{1}{2}\tilde{z}\right\}$.

Let $\mu = \hat{\mu} - \frac{\tilde{z}}{2n}\theta$, then $\tilde{z} = \frac{2n}{\theta}(\hat{\mu} - \mu)$ and $\left|\frac{d\tilde{z}}{d\mu}\right| = \frac{2n}{\theta}$.

Therefore

$$\begin{aligned} g(\mu|\theta) &= \tfrac{n}{\theta}\exp\left\{-\tfrac{n}{\theta}(\hat{\mu} - \mu)\right\} \qquad -\infty < \mu < \hat{\mu} \\ &= p(\mu|\theta, data) \end{aligned}$$

See Equation 2.5.

B Proof of Theorem 3.1

As before

$$f(\hat{\mu}_f|\mu, \theta) = \left(\frac{m}{\theta}\right)\exp\left\{-\frac{m}{\theta}(\hat{\mu}_f - \mu)\right\} \quad \hat{\mu}_f > \mu$$

and therefore

$$\tilde{f}(\hat{\mu}_f|\mu, data) = \int_0^\infty f(\hat{\mu}_f|\mu, \theta)\,\tilde{p}(\theta|\mu, data)\,d\theta.$$

Since

$$\tilde{p}(\theta|\mu, data) = \frac{\{n(\bar{x} - \mu)\}^n}{\Gamma(n)}\left(\frac{1}{\theta}\right)^{n+1}\exp\left\{-\frac{n}{\theta}(\bar{x} - \mu)\right\}$$

it follows that

$$\tilde{f}(\hat{\mu}_f|\mu, data) = \frac{n^{n+1}(\bar{x} - \mu)^n m}{\left[m(\hat{\mu}_f - \mu) + n(\bar{x} - \mu)\right]^{n+1}} \quad \hat{\mu}_f > \mu.$$

For $-\infty < \mu < x_{(1)}$,

$$\tilde{p}(\mu|data) = (n-1)\left(\hat{\theta}\right)^{n-1}\left(\frac{1}{\bar{x} - \mu}\right)^n$$

and

$$\begin{aligned} \tilde{f}(\hat{\mu}_f, \mu|data) &= \tilde{f}(\hat{\mu}_f|\mu, data)\,\tilde{p}(\mu|data) \\ &= \tfrac{n^{n+1}m(n-1)\left(\hat{\theta}\right)^{n-1}}{\left[m(\hat{\mu}_f - \mu) + n(\bar{x} - \mu)\right]^{n+1}} \end{aligned}.$$

Therefore

$$
\begin{aligned}
f\left(\hat{\mu}_f|data\right) &= \int_{-\infty}^{\hat{\mu}_f} f\left(\hat{\mu}_f,\mu|data\right)d\mu \quad -\infty<\hat{\mu}_f<x_{(1)}\\
&= \int_{-\infty}^{x_{(1)}} f\left(\hat{\mu}_f,\mu|data\right)d\mu \quad x_{(1)}<\hat{\mu}<\infty\\
&= \tilde{K}^*\left[\frac{1}{n\left(\bar{x}-\hat{\mu}_f\right)}\right]^n \quad -\infty<\hat{\mu}_f<x_{(1)}\\
&= \tilde{K}^*\left[\frac{1}{n\hat{\theta}+m\left(\hat{\mu}_f-x_{(1)}\right)}\right]^n \quad x_{(1)}<\hat{\mu}_f<\infty
\end{aligned}
$$

where

$$
\tilde{K}^*=\frac{n^n\left(n-1\right)m}{\left(n+m\right)}\left(\hat{\theta}\right)^{n-1}.
$$

C Proof of Theorem 3.3

From Equation 1.1 it follows that

$$
\hat{\theta}_f|\theta\sim\frac{\chi^2_{2m-2}}{2m}\theta
$$

which means that

$$
f\left(\hat{\theta}_f|\theta\right)=\left(\frac{m}{\theta}\right)^{m-1}\frac{\left(\hat{\theta}_f\right)^{m-2}\exp\left(-\frac{m}{\theta}\hat{\theta}_f\right)}{\Gamma\left(m-1\right)} \quad 0<\hat{\theta}_f<\infty \tag{C.1}
$$

The posterior distribution of θ (Equation 2.3) is

$$
\tilde{p}\left(\theta|data\right)=\frac{\left(n\hat{\theta}\right)^{n-1}}{\Gamma\left(n-1\right)}\left(\frac{1}{\theta}\right)^n\exp\left(-\frac{n}{\theta}\right) \quad 0<\theta<\infty.
$$

Therefore

$$
\begin{aligned}
\tilde{f}\left(\hat{\theta}_f|data\right) &= \int_0^\infty f\left(\hat{\theta}_f|\theta\right)p\left(\theta|\tilde{data}\right)d\theta\\
&= \frac{m^{m-1}}{\Gamma(m-1)}\frac{\left(n\hat{\theta}\right)^{n-1}}{\Gamma(n-1)}\left(\hat{\theta}_f\right)^{m-2}\int_0^\infty\left(\frac{1}{\theta}\right)^{m+n-1}\exp\left\{-\frac{1}{\theta}\left(m\hat{\theta}_f+n\hat{\theta}\right)\right\}d\theta\\
&= \frac{\Gamma(m+n-2)m^{m-1}\left(n\hat{\theta}\right)^{n-1}\left(\hat{\theta}_f\right)^{m-2}}{\Gamma(m-1)\Gamma(n-1)\left(m\hat{\theta}_f+n\hat{\theta}\right)^{m+n-2}} \quad 0<\hat{\theta}_f<\infty.
\end{aligned}
$$

D Proof of Corollary 3.4

$$
\hat{\theta}_f|\theta\sim\frac{\chi^2_{2m-2}}{2m}\theta
$$

and

$$
\theta|data\sim\hat{\theta}\frac{2n}{\chi^2_{2n-2}}.
$$

Therefore

$$\hat{\theta}_f|data \sim \frac{\chi^2_{2m-2}}{2m}\frac{2n}{\chi^2_{2n-2}}\hat{\theta},$$

$$\frac{1}{\hat{\theta}}\frac{2m}{2n}\frac{2n-2}{2m-2}\hat{\theta}_f \sim F_{2m-2;2n-2}$$

and

$$\hat{\theta}_f \sim \hat{\theta}\frac{n}{m}\frac{(m-1)}{(n-1)}F_{2(m-1);2(n-1)}.$$

E Proof of Theorem 6.1

Let

$$\hat{\theta} = \sum_{i=1}^{m}\hat{\theta}_i.$$

As mentioned in Section 6 (see also Krishnamoorthy and Mathew (2009)) that it is well known that

$$\hat{\theta}_i \sim \frac{\theta}{2n}\chi^2_{2(n-1)}$$

which means that

$$\hat{\theta} \sim \frac{\theta}{2n}\chi^2_{2m(n-1)}.$$

Therefore

$$f\left(\hat{\theta}|\theta\right) = \left(\frac{n}{\theta}\right)^{m(n-1)}\frac{\left(\hat{\theta}\right)^{m(n-1)-1}\exp\left(-\frac{n\hat{\theta}}{\theta}\right)}{\Gamma\left[m\left(n-1\right)\right]} = L\left(\theta|\hat{\theta}\right)$$

i.e, the likelihood function.

As before we will use as prior $p(\theta) \propto \theta^{-1}$.

The posterior distribution

$$\begin{aligned} p\left(\theta|\hat{\theta}\right) &= p\left(\theta|data\right) \propto L\left(\theta|\hat{\theta}\right)p\left(\theta\right) \\ &= \frac{\left(n\hat{\theta}\right)^{m(n-1)}}{\Gamma[m(n-1)]}\left(\frac{1}{\theta}\right)^{m(n-1)+1}\exp\left(-\frac{n\hat{\theta}}{\theta}\right) \end{aligned}$$

An Inverse Gamma distribution.

F Proof of Theorem 6.2

$$\hat{\theta}_f|\theta \sim \frac{\theta}{2n}\chi^2_{2(n-1)}.$$

Therefore

$$\begin{aligned}
f\left(\hat{\theta}_f|data\right) &= \int_0^\infty f\left(\hat{\theta}_f|\theta\right) p\left(\theta|data\right) d\theta \\
&= \int_0^\infty \left(\tfrac{n}{\theta}\right)^{n-1} \frac{\left(\hat{\theta}_f\right)^{n-2}\exp\left(-\frac{n\hat{\theta}_f}{\theta}\right)}{\Gamma(n-1)} \times \\
&\quad \frac{\left(n\hat{\theta}\right)^{m(n-1)}}{\Gamma[m(n-1)]}\left(\tfrac{1}{\theta}\right)^{m(n-1)+1}\exp\left(-\tfrac{n\hat{\theta}}{\theta}\right) d\theta \\
&= \frac{(n)^{n-1}\left(\hat{\theta}_f\right)^{n-2}\left(n\hat{\theta}\right)^{m(n-1)}}{\Gamma(n-1)\Gamma[m(n-1)]}\int_0^\infty \left(\tfrac{1}{\theta}\right)^{m(n-1)+n}\exp\left\{-\tfrac{n}{\theta}\left[\hat{\theta}_f+\hat{\theta}\right]\right\} d\theta \\
&= \frac{\Gamma[m(n-1)+n-1]\left(\hat{\theta}\right)^{m(n-1)}}{\Gamma(n-1)\Gamma[m(n-1)]}\frac{\left(\hat{\theta}_f\right)^{n-2}}{\left(\hat{\theta}_f+\hat{\theta}\right)^{m(n-1)+n-1}} \quad \hat{\theta}_f > 0
\end{aligned}$$

From this it follows that

$$\hat{\theta}_f|data \sim \frac{\hat{\theta}}{m}F_{2(n-1);2m(n-1)}$$

where

$$\hat{\theta} = \sum_{i=1}^{m}\hat{\theta}_i.$$

G Proof of Theorem 6.3

For given θ

$$\begin{aligned}
\psi\left(\theta\right) &= p\left(\hat{\theta}_f \le \tfrac{\hat{\theta}}{m}F_{2(n-1);2m(n-1)}\left(\beta\right)\right) \\
&= p\left(\tfrac{\theta}{2n}\chi^2_{2(n-1)} \le \tfrac{\hat{\theta}}{m}F_{2(n-1);2m(n-1)}\left(\beta\right)\right) \quad \text{given } \theta \\
&= p\left(\frac{2n\hat{\theta}}{\chi^2_{2m(n-1)}}\frac{\chi^2_{2(n-1)}}{2n} \le \tfrac{\hat{\theta}}{m}F_{2(n-1);2m(n-1)}\left(\beta\right)\right) \quad \text{given } \chi^2_{2m(n-1)} \\
&= p\left(\chi^2_{2(n-1)} \le \tfrac{\chi^2_{2m(n-1)}}{m}F_{2(n-1);2m(n-1)}\left(\beta\right)\right) \quad \text{given } \chi^2_{2m(n-1)} \\
&= \psi\left(\chi^2_{2m(n-1)}\right)
\end{aligned}$$