Bayesian Results for the Process Capability Indices C_{pl}, C_{pu} and C_{pk} and Control Chart for the C_{pk} Index

R. van Zyl[1], A.J. van der Merwe[2]

[1]Quintiles International, ruaanvz@gmail.com

[2]University of the Free State

UNIVERSITY OF THE
FREE STATE
UNIVERSITEIT VAN DIE
VRYSTAAT
YUNIVESITHI YA
FREISTATA

UFS·UV
NATURAL AND
AGRICULTURAL SCIENCES
NATUUR- EN
LANDBOUWETENSKAPPE
MATHEMATICAL STATISTICS
AND ACTUARIAL SCIENCE
WISKUNDIGE STATISTIEK
EN AKTUARIËLE WETENSKAP

Abstract

Process capability indices have been widely used in the manufacturing industry. They measure the ability of a manufacturing process to produce items that meet certain specifications. A capability index relates the voice of the customer (specification limits) to the voice of the process. A large value of the index indicates that the current process is capable of producing items (parts, tablets) that will meet or exceed the customers' requirements. Capability indices are convenient because they reduce complex information about the process to a single number and measure relative variability similar to the coefficient of variation.

This paper developed a Bayesian method to analyze capability indices. Multiple testing strategies will be implemented and results will be compared to the frequentist results. Control charts were further developed using the Bayesian approach for C_{pk}.

Keywords: capability indices, Bayes, credibility intervals, Tukey, control charts, run lengths

1 Introduction

Process capability indices have been widely used in the manufacturing industry. They measure the ability of a manufacturing process to produce items that meet certain specifications. A capability index relates the voice of the customer (specification limits) to the voice of the process. A large value of the index indicates that the current process is capable of producing items (parts, tablets) that will meet or exceed the customers' requirements. Capability indices are convenient because they reduce complex information about the process to a single number and measure relative variability similar to the coefficient of variation.

Application examples include the manufacturing of semiconductor products (Hoskins, Stuart, and Taylor (1988)), jet-turbine engine components (Hubele, Shahriari, and Cheng (1991)), wood products (Lyth and Rabiej (1995)), audio speaker drivers (Chen and Pearn (1997)) and many others.

There is a need to understand and interpret process capability indices. In the literature on statistical quality control there have been some attempts to study the inferential aspects of these indices. Most of the existing works in this area has been devoted to classical frequentist large sample theory.

As mentioned by Pearn and Wu (2005) a point estimate of the index is not very useful in making reliable decisions. An interval estimation approach is in fact more appropriate and widely accepted but the frequency distributions of these estimators are often very complicated which means that the calculation of exact confidence intervals will be difficult.

An alternative approach to the problem of making inference about capability indices is the Bayesian approach. As it is well known in the Bayesian approach the information contained in the prior is combined with the likelihood to obtain the posterior distribution of the parameters. Inferences about the unknown parameters are based on the posterior distribution.

2 Definitions and Notations

Four of the commonly used capability indices are:

$$C_p = \frac{u - l}{6\sigma},$$

$$C_{pu} = \frac{u - \mu}{3\sigma},$$

$$C_{pl} = \frac{\mu - l}{3\sigma}$$

and

$$C_{pk} = min\left(C_{pu}, C_{pl}\right).$$

C_{pk} is the normalized distance between the process mean and its closest specification limit. It can easily be verified that $C_{pk} = C_p\left(1 - w\right)$ where $w = \frac{2|m-\mu|}{u-l}$ and $m = \frac{u+l}{2}$ is the midpoint of the specification limits (u and l). Thus, C_{pk} modifies C_p by a standardized measure w of non-centrality of the process and $C_{pk} = C_p$ if and only if the process is centered at m.

The larger the value of C_{pk}, the more capable is the process. In general, if the value of a process capability index is greater than 1.0 the process is said to be capable. According to Niverthi and Dey (2000), the thrust these days in the manufacturing industry is to achieve a C_{pk} value of at least 1.33. The definition of C_{pk} includes as special case those processes for which only one limit exists, by letting either $l \to -\infty$ or $u \to \infty$, in which case it reduces to the appropriate standardized measure.

Let $y_1, y_2, \ldots, y_n$ be an independent sample from a manufacturing process. In this paper it will be assumed that the y_i $(i = 1, 2, \ldots, n)$ are independent, identically normally distributed random variables with mean μ and variance σ^2. Since both μ and σ^2 are unknown and no prior information is available, the conventional non-informative, Jeffreys' prior

$$p\left(\mu, \sigma^2\right) \propto \sigma^{-2} \tag{2.1}$$

will be specified for μ and σ in this section. Using (2.1), it is well known (see for example Zellner (1971)) that the conditional posterior density function of μ is normal:

$$\mu|\sigma^2, \underline{y} \sim N\left(\bar{y}, \frac{\sigma^2}{n}\right) \tag{2.2}$$

and in the case of the variance component σ^2, the posterior density function is given by

$$p\left(\sigma^2|\underline{y}\right) = K\left(\sigma^2\right)^{-\frac{1}{2}(n+1)} \exp\left\{-\frac{1}{2}\frac{(n-1)\,s^2}{\sigma^2}\right\} \quad \sigma^2 > 0 \tag{2.3}$$

an Inverted Gamma density function where $\underline{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}'$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ is the sample mean, $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2$ is the sample variance and

$$K = \left\{\frac{(n-1)\,s^2}{2}\right\}^{\frac{1}{2}(n-1)} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \tag{2.4}$$

is a normalizing constant.

From (2.3) it follows that

$$k = \frac{(n-1)\,s^2}{\sigma^2} \sim \chi^2_{n-1} = \chi^2_v \tag{2.5}$$

for a given s^2.

As mentioned these indices are used in process evaluation. From a Bayesian point of view the posterior distributions are of importance. One of the aims of this paper is therefore to derive the exact and in some cases the conditional posterior distributions of the indices. The method proposed by Ganesh (2009) for multiple testing will be applied using a Bayesian procedure for C_{pl}, C_{pu} and C_{pk} to determine whether significant differences between four suppliers exist. A Bayesian control chart for C_{pk} will also be implemented.

An estimated index will be denoted by "hat" ($\hat{}$). For example $\hat{C}_p = \frac{u-l}{6s}$, $\hat{C}_{pl} = \frac{\bar{y}-l}{3s}$, $\hat{C}_{pu} = \frac{u-\bar{y}}{3s}$ and $\hat{C}_{pk} = min\left(\hat{C}_{pu}, \hat{C}_{pl}\right)$.

The following theorems will now be proved.

3 The Posterior Distribution of the Lower Process Capability Index $C_{pl} = \frac{\mu-l}{3\sigma}$

Theorem 3.1. *The posterior distribution of $t = C_{pl}$ is given by*

$$p\left(t|\tilde{t}\right) = \frac{3\sqrt{n}\exp\left\{-\frac{9nt^2}{2}\right\}}{\Gamma\left(\frac{v}{2}\right)\sqrt{2\pi}} \sum_{j=0}^{\infty} \left(\frac{9nt\tilde{t}}{\sqrt{v}}\right)^j \frac{1}{j!} \frac{\Gamma\left(\frac{v+j}{2}\right) 2^{\frac{1}{2}j}}{\left(1+\frac{9n}{v}\tilde{t}^2\right)^{\frac{v+j}{2}}} \qquad -\infty < t < \infty \tag{3.1}$$

where

$$\tilde{t} = \frac{\bar{y}-l}{3s} = \hat{C}_{pl}$$

and

$$v = n-1.$$

Proof. The proof is given in the Mathematical Appendices to this paper. □

Note:

Chou and Owen (1989) derived the distribution of $\tilde{t}$ which is given by

$$f\left(\tilde{t}|t\right) = \frac{3\sqrt{n}\exp\left\{-\frac{9nt^2}{2}\right\}}{\sqrt{v}\sqrt{2\pi}\Gamma\left(\frac{v}{2}\right)} \sum_{j=0}^{\infty} \left(\frac{9n\tilde{t}t}{\sqrt{v}}\right)^j \frac{1}{j!} \frac{\Gamma\left(\frac{v+j+1}{2}\right) 2^{\frac{1}{2}j}}{\left(1+\frac{9n}{v}\tilde{t}^2\right)^{\frac{v+j+1}{2}}}. \tag{3.2}$$

Equation (3.2) is a non-central t distribution with v degrees of freedom and non-centrality parameter δ where $\delta^2 = 9nt^2$.

4 The Posterior Distribution of $C_{pk} = min\left(C_{pl}, C_{pu}\right)$

When both specification limits are given, the C_p and C_{pk} indices can be used where

$$C_{pk} = min\left(C_{pl}, C_{pu}\right).$$

Unlike C_p, C_{pk} depends on both μ and σ. The C_{pk} index has been used in Japan and in the U.S. automotive companies (see Kane (1986)) and Chou and Owen (1989).

In Theorem 4.1 the posterior distribution of $c = C_{pk}$ will be derived.

Theorem 4.1. *The posterior distribution of $c = C_{pk}$ is given by*

$$p\left(c|y\right) = \frac{3\sqrt{n}}{\sqrt{2\pi}} \int_{\frac{c^2 v}{\tilde{b}^2}}^{\infty} \left\{ \exp\left(-\frac{9n}{2} \left[c - t^* \sqrt{\frac{k}{v}} \right]^2 \right) + \exp\left(-\frac{9n}{2} \left[c - \tilde{t} \sqrt{\frac{k}{v}} \right]^2 \right) \right\} \frac{1}{2^{\frac{v}{2}} \Gamma\left(\frac{v}{2}\right)} k^{\frac{v}{2}-1} \exp\left(-\frac{k}{2}\right) dk \tag{4.1}$$

where

$$v = n - 1,$$

$$t^* = \hat{C}_{pu} = \frac{u - \bar{y}}{3s},$$

$$\tilde{t} = \hat{C}_{pl} = \frac{\bar{y} - l}{3s}$$

and

$$\tilde{b} = \hat{C}_p = \frac{u - l}{6s}.$$

Proof. The proof is given in the Mathematical Appendices to this paper. □

5 Example: Piston Rings for Automotive Engines (Polansky (2006))

Consider a company with $N = 4$ contracted suppliers representing the four processes that produce piston rings for automobile engines studied by Chou (1994). The edge width of a piston ring after the preliminary disk grind is a very important quality characteristic in automobile engine manufacturing. The lower and upper specification limits of the quality characteristic are $l = 2.6795mm$ and $u = 2.7205mm$ respectively Four potential suppliers (Supplier 1 to Supplier 4) for such rings are under consideration by one quality control manager. Samples of size $n_1 = 50$, $n_2 = 75$, $n_3 = 70$ and $n_4 = 75$ are taken from the manufacturing processes of the suppliers. A summary of the results from the samples, $\hat{C}_{pl}$, $\hat{C}_{pu}$, $\hat{C}_{pk}$ values and other statistics are given in Table 5.1.

Table 5.1: $\hat{C}_{pl}$, $\hat{C}_{pu}$, and $\hat{C}_{pk}$ Values for the Four Suppliers

Supplier (i)	1	2	3	4
Sample Size (n_i)	50	75	70	75
Estimated Mean $(\bar{y}_i)$	2.7048	2.7019	2.6979	2.6972
Estimated Standard Deviation (s_i)	0.0034	0.0055	0.0046	0.0038
$\hat{C}_{pl}^{(i)} = \frac{\bar{y}_i - l}{3s_i}$	2.4804	1.3576	1.3333	1.5526
$\hat{C}_{pu}^{(i)} = \frac{u - \bar{y}_i}{3s_i}$	1.5392	1.1273	1.6377	2.0439
$\hat{C}_{pk}^{(i)} = min\left(\hat{C}_{pl}^{(i)}, \hat{C}_{pu}^{(i)}\right)$	1.5392	1.1273	1.3333	1.5526

Looking at Table 5.1, it is clear that Suppliers 4 and 1 give the two largest values for $\hat{C}_{pl}$, $\hat{C}_{pu}$ and $\hat{C}_{pk}$, suggesting that they are the most capable. This may be because they seem to have the smallest variation within the specified range. They therefore represent the best two choices of suppliers. Suppliers 3 and 2 are not as capable as the former because of their greater variability. Because the estimated $\hat{C}_{pk}$ index for Supplier 1 is close to that of Supplier 4 we might feel that that the difference in capability of the processes between these suppliers is not significant. The same statement may hold true of Suppliers 2 and 3. Statistical methods for the comparison of the suppliers' process capability indices are required for the quality control manager to draw intelligent conclusion from this data. A Bayesian simulation procedure will be considered to determine which processes are significantly different from one another. The potential performance of the proposed method will be compared with the permutation approach by Polansky (2006).

Before discussion the simulation procedure the posterior distributions of the capability indices will be looked at.

In the last part of this paper control limits will be calculated for future capability indices. In Figure 5.1 the posterior distributions of C_{pk} are illustrated using Equation (4.1) and numerical integration.

Figure 5.1: Posterior Distributions of C_{pk}

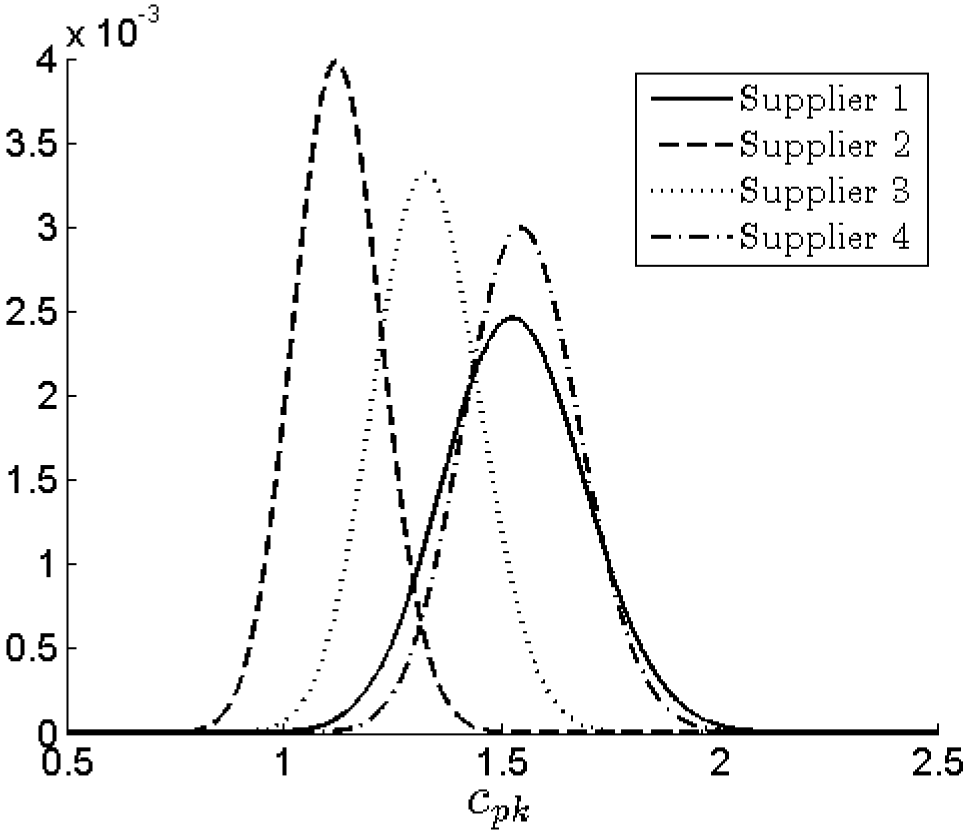


Table 5.2: Posterior Means and Variances

	Supplier 1	Supplier 2	Supplier 3	Supplier 4
Posterior Mean	1.5314	1.1234	1.3285	1.5474
Posterior Variance	0.0263	0.0100	0.0144	0.0177

From Table 5.2 it can be seen that the posterior means are for all practical purposes the same as the $\hat{C}_{pk}$ values given in table 5.1. Further inspection of Figure 5.1 and Table 5.2 shows that Suppliers 4 and 1 have the largest posterior means, suggesting they are the most capable. In the next section a simple Bayesian solution to the problem of constructing simultaneous credibility intervals for the capability indices will be discussed.

6 Simultaneous Credibility Intervals

The method proposed by Ganesh (2009) can be compared to multiple testing, also referred to as the multiple comparison problem. In multiple testing, the objective is to control the family wise error rate. Similarly in his paper, Ganesh control the simultaneous coverage rate. If the interest is in constructing simultaneous credibility intervals for all pairwise differences, a Bayesian version of Tukey's simultaneous confidence intervals can be used. Define

$$T^{(2)} = \underset{l}{max}\left\{\left(C_{pk}^{(l)} - E\left(C_{pk}^{(l)}|\underline{y}\right)|\underline{y}\right)\right\} - \underset{l}{min}\left\{C_{pk}^{(l)} - E\left(C_{pk}^{(l)}|\underline{y}\right)|\underline{y}\right\}$$

where $T_{\alpha}^{(2)}$ is the upper α percentage point of the distribution of $T^{(2)}$. Simultaneous $100\,(1-\alpha)\,\%$ credibility intervals for all pairwise differences are given by

$$E\left(C_{pk}^{(i)}|\underline{y}\right) - E\left(C_{pk}^{(j)}|\underline{y}\right) \pm T_{\alpha}^{(2)} \quad i = 1, 2, \ldots, 4;\ j = 1, 2, \ldots, 4;\ i \neq j$$

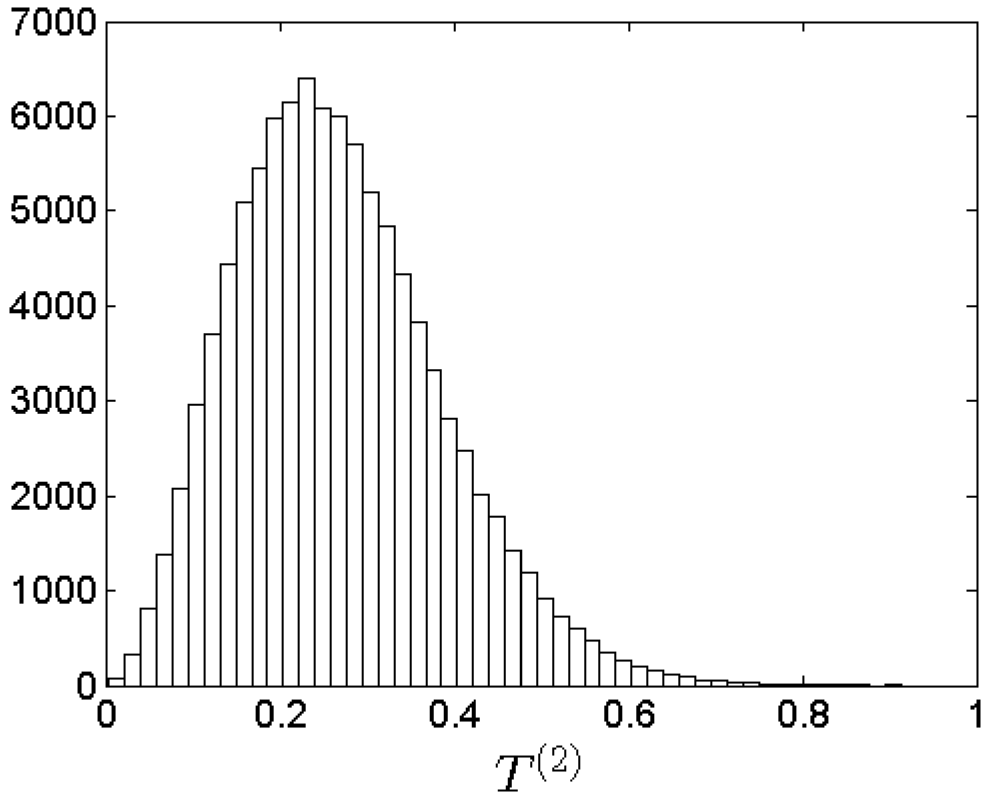
100,000 Monte Carlo simulations were used to calculate $E\left(C_{pk}^{(i)}|\underline{y}\right)$, $E\left(C_{pk}^{(j)}|\underline{y}\right)$ and $T_{\alpha}^{(2)}$.

The simulation procedure is as follows:

1. Simulate k from a χ_{n-1}^{2} distribution.
2. Calculate $\sigma_i^{2*} = \frac{(n-1)s_i^2}{k}$ (Equation (2.5)) where (*) indicates a simulated value ($i = 1, 2, \ldots, 4$).
3. $\sigma_i^{*} = \sqrt{\sigma_i^{2*}}$
4. By using the fact that $\mu_i|\sigma_i^2, \bar{y}_i \sim N\left(\bar{y}_i, \frac{\sigma_i^2}{n}\right)$ (Equation (2.2)) simulate μ_i^{*}.
5. From the definition of the capability index it follows that $C_{pk}^{(i)}$ can be simulated as $C_{pk}^{(i)*} = min\left(\frac{u-\mu_i^*}{3\sigma_i^*}, \frac{\mu_i^*-l}{3\sigma_i^*}\right)$.
6. Repeat steps 1 to 5 $\tilde{l}$ times. As mentioned, for this example $\tilde{l} = 100,000$.

In Figure 6.1 the posterior distribution of $T^{(2)}$ is given and in Table 6.1 credibility intervals for differences in C_{pk} are given using Ganesh's method.

Figure 6.1: Distribution of $T^{(2)}$



$T^{(2)}_{0.05} = 0.4823$, $T^{(2)}_{0.1} = 0.4279$ and $T^{(2)}_{0.15} = 0.3915$

Table 6.1: Credibility Intervals for Differences in C_{pk} - Ganesh Method

$E\left(C_{pk}^{(i)}\vert\underline{y}\right) - E\left(C_{pk}^{(j)}\vert\underline{y}\right)$	95% Interval	90% Interval	87.47% Interval
Supplier 1 - Supplier 2	(-0.0734;0.8915)	(-0.0187;0.8371)	**(0;0.8155)**
Supplier 1 - Supplier 3	(-0.2779;0.6867)	(-0.2234;0.6323)	(-0.2058;0.6097)
Supplier 1 - Supplier 4	(-0.4971;0.4675)	(-0.4427;0.4131)	(-0.4245;0.3910)
Supplier 2 - Supplier 3	(-0.6871;0.2775)	(-0.6326;0.2231)	(-0.6136;0.2019)
Supplier 2 - Supplier 4	(-0.9063;0.0583)	(-0.8519;0.0039)	**(-0.8323;-0.0168)**
Supplier 3 - Supplier 4	(-0.7016;0.2630)	(-0.6471;0.2086)	(-0.6265;0.1890)

For solving the supplier problem Polansky (2006) used multiple comparison techniques in conjunction with permutation tests. The multiple comparisons tests used were:

a. The Bonferonni method, which adjusts the significance levels of the pair wise tests.

b. The protected multiple comparison method, which requires that an omnibus test of equality between all of the process capability indices be rejected before pair-wise tests are performed and does not require adjustment of the significance level of the pair-wise tests.

Polansky (2006) came to the conclusion that at the 5% significance level suppliers 1,2 and 4 have process capabilities that are not significantly different. Similarly, suppliers 2 and 3 are not significantly different from one another, but supplier 2 is significantly different from Supplier 1 and 4.

According to Table 6.1 it is only at significance level of 12.5% that the Bayesian procedure shows a significant difference between Supplier 2 and Suppliers 1 and 4. To see if Ganesh (2009) version of Tukey's simultaneous confidence intervals is somewhat conservative, the following simulation study has been conducted to evaluate the coverage probability and power of the Bayesian hypothesis testing procedure.

I. a. Assume that $y \sim N(\mu_1, \sigma_1^2)$ where $\mu_1 = 2.7048$ and $\sigma_1^2 = (0.0034)^2$. The parameters μ_1 and σ_1^2 are obtained from the sample statistics of Supplier 1.

b. Simulate the sufficient statistics $\bar{y}_i \sim N\left(\mu_1, \frac{\sigma_1^2}{n_1}\right)$ and $(n_1 - 1) s_i^2 \sim \sigma_1^2 \chi^2_{n_1 - 1}$ to represent a data set for the four suppliers where $n_1 = 50$ and $i = 1, 2, 3, 4$.

c. By doing $\tilde{l} = 10000$ simulations $T^{(2)}_{0.05}$ can be calculated for our first dataset as well as the credibility intervals as described in Section 6.

d. If any one of the six credibility intervals do not contain zero, the null hypothesis

$$H_0 : C^{(1)}_{pk} = C^{(2)}_{pk} = C^{(3)}_{pk} = C^{(4)}_{pk}$$

will be rejected. Rejection of H_0 when it is true is called a Type I error.

e. Steps (a) - (d) are replicated $l = 20,000$ times with $\mu_1 = 2.7048$, $\sigma_1^2 = (0.0034)^2$ and $n_1 = 50$ and the estimated Type I error $= \frac{1008}{20000} = 0.0504$ which corresponds well with $\alpha = 0.05$. It means that for 1008 datasets one or more of the six credibility intervals did not contain zero.

II. Assume now that $y \sim N(\mu_2, \sigma_2^2)$ where $\mu_2 = 2.7019$, $\sigma_2^2 = (0.0054)^2$ and $n_2 = 75$. The parameter values are that of the sample statistics of the second supplier. Repeat steps I (a) - I (e) and also for Suppliers 3 and 4.

In Table 6.2 the estimated Type I errors for the four cases are given.

Table 6.2: Estimated Type I Error for Different Parameter Combinations and Sample Sizes

n	μ	σ	Type I Error
50	2.7048	0.0034	0.0504
75	2.7019	0.0054	0.0483
70	2.6979	0.0046	0.0521
75	2.6972	0.0038	0.0507

The average Type I error = 0.0504 which as mentioned corresponds well with $\alpha = 0.05$. It therefore does not seem that Ganesh Bayesian version of Tukey's simultaneous confidence interval is too conservative.

7 Type II Error of Ganesh Bayesian Method

Acceptance of H_0 when it is false is called a Type II error. In Table 7.1 the sample statistics of Table 5.1 are used as parameter values.

Table 7.1: C_{pk} Values for the Four Suppliers

Supplier (i)	1	2	3	4
Sample Size (n_i)	50	75	70	75
Mean (μ_i)	2.7048	2.7019	2.6979	2.6972
Standard Deviation (σ_i)	0.0034	0.0054	0.0046	0.0038
$\hat{C}^{(i)}_{pk}$	1.5392	1.1273	1.3333	1.5526

It is clear from Table 7.1 that the C_{pk} parameters values are all different. To get an estimate of the Type II error 10 000 data sets were generated with sample sizes as shown in Table 7.1. For each dataset $T^{(2)}_{\alpha}$

was calculated from 10,000 Monte Carlo simulations. The Type II error was estimated by observing the number of times that H_0 was accepted, i.e., the number of times that all six credibility intervals contain zero. The process was repeated 5 times and the following estimates of the Type II error were obtained: 0.4240, 0.4163, 0.4212 and 0.4182. The average Type II error estimate is therefore 0.42184 and is the result of 50,000 datasets. The power of the Bayesian procedure $= 1 - 0.42184 = 0.57816$.

8 Posterior Distributions of C_{pl} and C_{pu}

It might be of interest to also look at the posterior distributions of $C_{pl} = \frac{\mu - l}{3\sigma}$ and $C_{pu} = \frac{u-\mu}{3\sigma}$. The posterior distribution of C_{pl} is given in Equation (3.1) and can be used for illustration purposes. A much easier way to obtain the posterior distribution is to simulate a large number of conditional posterior distributions. The average of these conditional distributions is then the unconditional posterior distribution of C_{pl}. This procedure is called the Rao-Blackwell method.

In Figures 8.1 and 8.2 the posterior distributions of C_{pl} and C_{pu} are displayed.

Figure 8.1: Posterior Distributions of C_{pl}

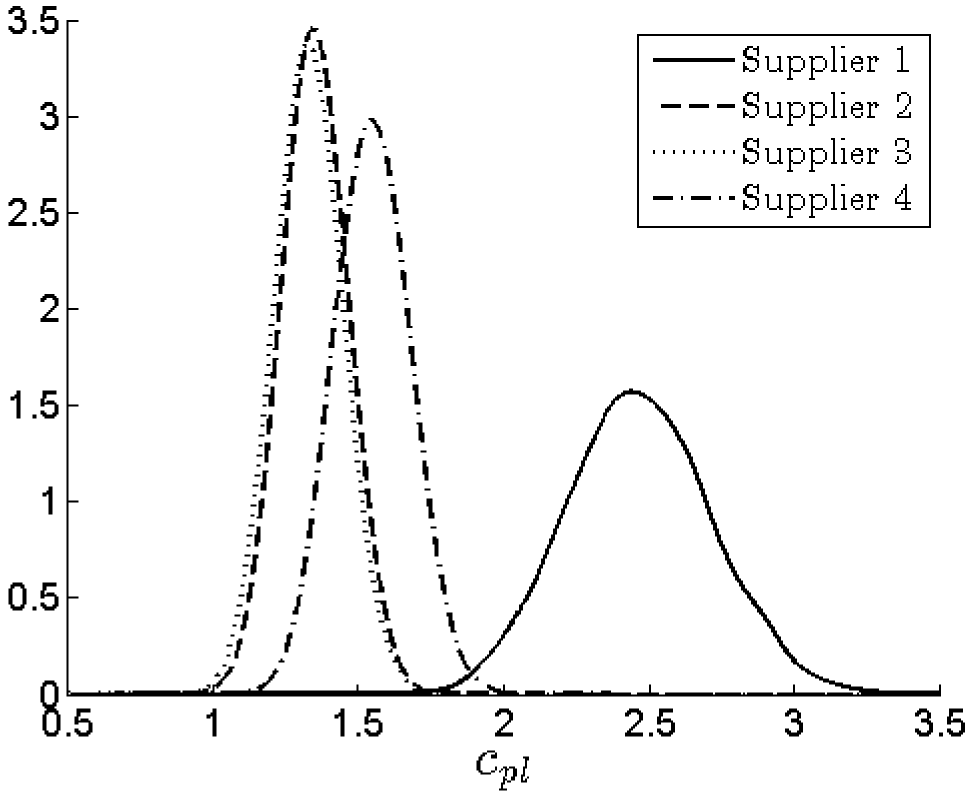
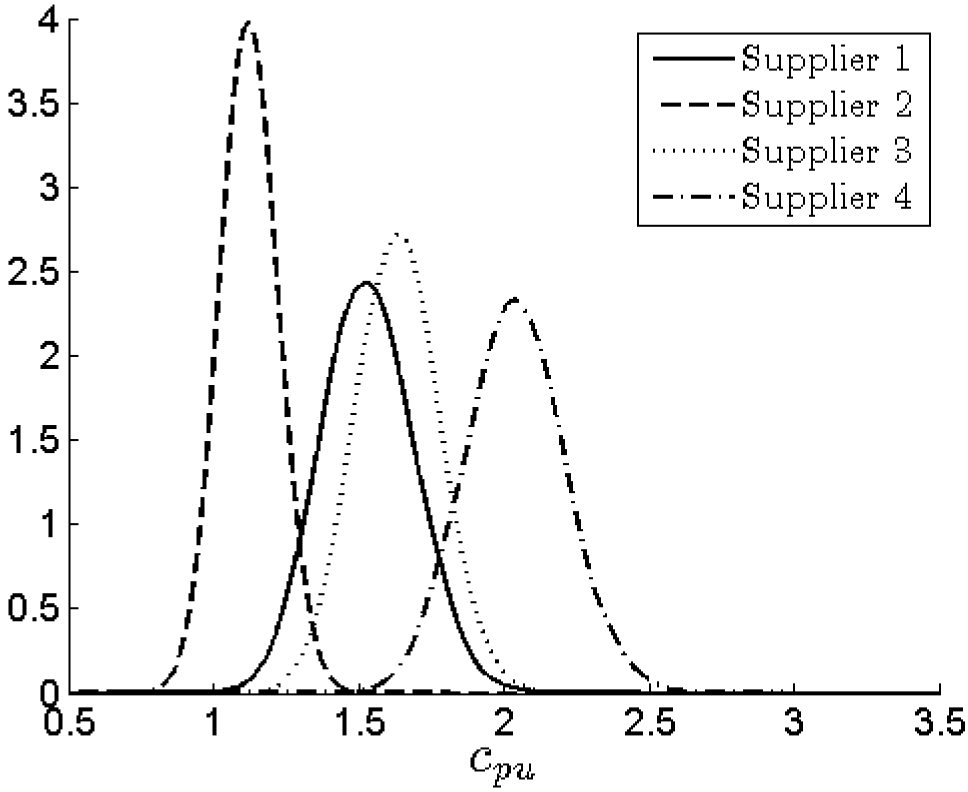


Figure 8.2: Posterior Distributions of C_{pu}



In Table 8.1 the posterior means of C_{pl} and C_{pu} are given for the four suppliers and in Table 8.2 the 95% credibility intervals for the differences between suppliers are given using Ganesh method.

Table 8.1: Posterior Means of C_{pl} and C_{pu}

Supplier	1	2	3	4
C_{pl}	2.4920	1.3615	1.3377	1.5585
C_{pu}	1.5460	1.1303	1.6431	2.0521

Table 8.2: 95% Credibility Intervals for Differences between Suppliers

	C_{pl}	C_{pu}
Supplier 1 - Supplier 2	**(0.4910;1.7700)**	(-0.1281;0.9595)
Supplier 1 - Supplier 3	**(0.5148;1.7938)**	(-0.6408;0.4467)
Supplier 1 - Supplier 4	**(0.2940;1.5729)**	(-1.0498;0.0377)
Supplier 2 - Supplier 3	(-0.6157;0.6633)	(-1.0565;0.0310)
Supplier 2 - Supplier 4	(-0.8365;0.4424)	**(-1.4655;-0.3780)**
Supplier 3 - Supplier 4	(-0.8603;0.4187)	(-0.9527;0.1348)

According to the C_{pl} credibility interval Supplier 1 is significantly different from Suppliers 2, 3 and 4. The other suppliers do not differ significantly from each other. Inspection of the C_{pu} intervals shows that there is a significant difference between Suppliers 2 and 4.

9 The Predictive Distribution of a Future Sample Capability Index, $\hat{C}_{pk}^{(f)}$

To obtain a Bayesian control chart for the capability index C_{pk} the predictive distribution must first be derived.

Consider a future sample of m observations from the $N\left(\mu,\sigma^2\right)$ population, $y_{1f}, y_{2f}, \ldots, y_{mf}$. The future sample mean is defined as $\bar{y}_f = \frac{1}{m}\sum_{j=1}^{m} y_{jf}$ and a future sample variance by $s_f^2 = \frac{1}{m-1}\sum_{j=1}^{m}\left(y_{jf}-\bar{y}_f\right)^2$. A future sample capability index is therefore defined as

$$\hat{C}_{pk}^{(f)} = min\left(\hat{C}_{pu}^{(f)}, \hat{C}_{pl}^{(f)}\right)$$

where

$$\hat{C}_{pu}^{(f)} = \frac{u-\bar{y}_f}{3s_f}$$

and

$$\hat{C}_{pl}^{(f)} = \frac{\bar{y}_f - l}{3s_f}.$$

By using the results given in Smit and Chakraborti (2009) or by using similar theoretical derivations as in Theorem 4.1 it can be shown that the conditional predictive distribution of $\hat{C} = \hat{C}_{pk}^{(f)}$ is given by

$$f\left(\hat{C}|\mu,\sigma^2\right) = 3\sqrt{\frac{m}{m-1}}\int_0^{\frac{m-1}{\hat{C}^2}C_p^2}\left\{\phi\left(3\sqrt{m}\left[C-2C_p+\sqrt{\frac{x}{m-1}}\hat{C}\right]\right)+\phi\left(3\sqrt{m}\left[C-\sqrt{\frac{x}{m-1}}\hat{C}\right]\right)\right\}\sqrt{x}f\left(x\right)dx \qquad (9.1)$$

where

$$f\left(x\right) = \frac{1}{2^{\frac{m-1}{2}}\Gamma\left(\frac{m-1}{2}\right)}x^{\frac{1}{2}(m-1)-1}\exp\left(-\frac{x}{2}\right),$$

$$\phi\left(z\right) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right),$$

$$C = C_{pk} = min\left(\frac{u-\mu}{3\sigma}, \frac{\mu-l}{3\sigma}\right)$$

and

$$C_p = \frac{u-l}{6\sigma}.$$

The unconditional predictive distribution $f\left(\hat{C}|data\right)$ can be obtained in the following way:

i. Simulate σ^2 and μ from their joint posterior distribution and calculate C and C_p. Since $\sigma^2|data \sim \frac{(n-1)S^2}{\chi_{n-1}^2}$ and $\mu|\sigma^2, data \sim N\left(\bar{y}, \frac{\sigma^2}{n}\right)$, μ and σ^2 can easily be simulated. Let us call these simulated values μ_1 and σ_1^2.

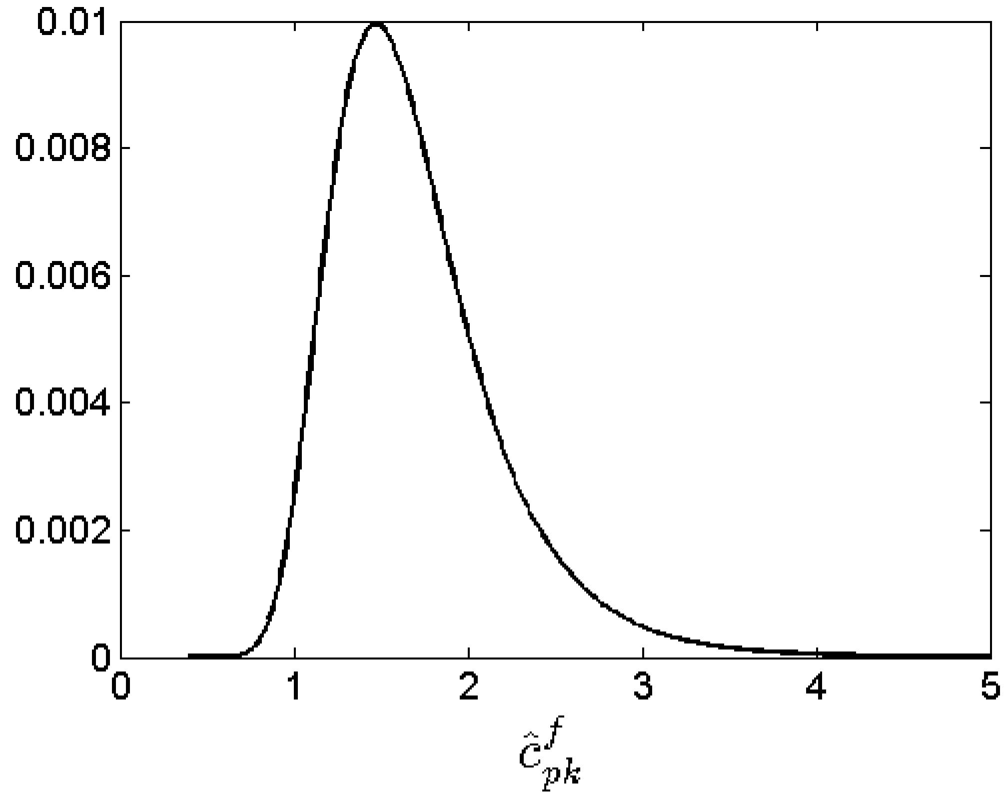
ii. Substitute μ_1 and σ_1^2 in Equation (9.1) and do the numerical integration to obtain $f\left(\hat{C}|\mu_1,\sigma_1^2\right)$.

iii. Repeat (i) and (ii) l times to get $f\left(\hat{C}|\mu_1,\sigma_1^2\right), f\left(\hat{C}|\mu_2,\sigma_2^2\right), \ldots, f\left(\hat{C}|\mu_l,\sigma_l^2\right)$. The unconditional predictive distribution $f\left(\hat{C}|data\right)$ is the average of the conditional predictive distributions (Rao-Blackwell method).

9.1 Example

Consider the following sample values: $n = 75$, $\bar{y} = 2.6972$ and $s = 0.0038$. These sample values are the statistics for Supplier 4. In Figure 9.1 the predictive distribution of $\hat{C} = \hat{C}_{pk}^{(f)}$ for $m = 10$ future observations are given.

Figure 9.1: $f\left(\hat{C}_{pk}^{(f)}|data\right)$



$$Mean\left(\hat{C}_{pk}^{(f)}\right) = 1.6870;\ Median\left(\hat{C}_{pk}^{(f)}\right) = 1.598;\ Mode\left(\hat{C}_{pk}^{(f)}\right) = 1.456;\ Var\left(\hat{C}_{pk}^{(f)}\right) = 0.2432$$
$$95\%\ Equal - tail\ interval = (0.9936; 2.8954)\,,\ length = 1.9018$$
$$95\%\ HPD\ interval = (0.8790; 2.6360)\,,\ length = 1.7570$$
$$p\left(\hat{C}_{pk}^{(f)} > 3.923\right) = 0.0027$$
$$p\left(\hat{C}_{pk}^{(f)} > 4.263\right) = 0.00135$$
$$p\left(\hat{C}_{pk}^{(f)} < 0.7905\right) = 0.00135$$

10 Distribution of the Run-length and Average Run-length

Assuming that the process remains stable, the predictive distribution can be used to derive the distribution of the run-length and average run-length. From Figure 9.1 it follows that for a 99.73% two-sided control chart the lower control limit is $LCL = 0.7905$ and the upper control limit is $UCL = 4.263$. If a future capability index is smaller than 0.7905 or larger than 4.263, it falls in the rejection region and it is said that the control chart signals. The run-length is defined as the number of future $C_{pk}^{(f)}$ indices (r) until the control chart signals for the first time (Note that r does not include that $C_{pk}^{(f)}$ index when the control chart signals). Given μ and σ^2 and a stable Phase I process, the distribution of the run-length r

is Geometric with parameter

$$\psi\left(\mu,\sigma^2\right) = \int_{R(\beta)} f\left(\hat{C}_{pk}^{(f)}|\mu,\sigma^2\right) d\hat{C}_{pk}^{(f)}$$

where $f\left(\hat{C}_{pk}^{(f)}|\mu,\sigma^2\right)$ is defined in Equation (9.1), i.e., the distribution of $\hat{C}_{pk}^{(f)}$ given that μ and σ^2 are known and $R(\beta)$ represents those values of $C_{pk}^{(f)}$ that are smaller than LCL and larger than UCL. The values of μ and σ^2 are however unknown and the uncertainty of these parameters are described by the joint posterior distribution $p\left(\mu,\sigma^2|data\right) = p\left(\mu|\sigma^2, data\right) p\left(\sigma^2|data\right)$ (Equations (2.2) and (2.3)).

By simulating μ and σ^2 from $p\left(\mu,\sigma^2|data\right)$ the probability density function of $f\left(\hat{C}_{pk}^{(f)}|\mu,\sigma^2\right)$ as well as the parameter $\psi\left(\mu,\sigma^2\right)$ can be obtained. This must be done for each future sample. In other words, for each future sample μ and σ^2 must first be simulated from $p\left(\mu,\sigma^2|data\right)$ and then $\psi\left(\mu,\sigma^2\right)$ calculated. Therefore by simulating all possible combinations of μ and σ^2 from their joint posterior distribution a large number of $\psi\left(\mu,\sigma^2\right)$ values can be obtained. Also a large number of Geometric distributions, i.e., a large number of run-length distributions each with a different parameter value $\left(\psi\left(\mu_1,\sigma_1^2\right), \psi\left(\mu_2,\sigma_2^2\right), \ldots, \psi\left(\mu_l,\sigma_l^2\right)\right)$ can be obtained.

As mentioned the run-length r for given μ and σ^2 is geometrically distributed with mean

$$E\left(r|\mu,\sigma^2\right) = \frac{1-\psi\left(\mu,\sigma^2\right)}{\psi\left(\mu,\sigma^2\right)}$$

and variance

$$Var\left(r|\mu,\sigma^2\right) = \frac{1-\psi\left(\mu,\sigma^2\right)}{\psi^2\left(\mu,\sigma^2\right)}.$$

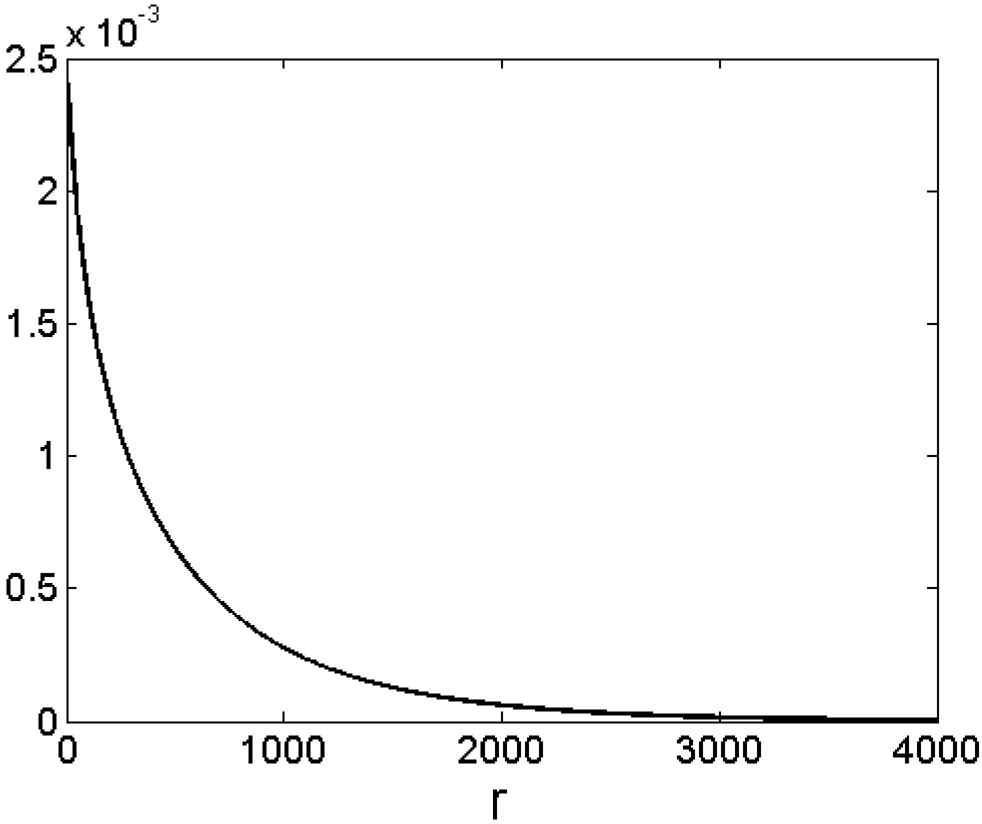
The unconditional moments $E\left(r|data\right)$, $E\left(r^2|data\right)$ and $Var\left(r|data\right)$ can therefore be obtained by simulation or numerical integration. For further details see Menzefricke (2002, 2007, 2010b,a) .

The mean of the predictive distribution of the run-length for the 99.73% two-sided control limits is $E\left(r|data\right) = 482.263$ somewhat larger than the 370 that one would have expected if $\beta = 0.0027$. The median on the other hand is less than 370, $Median\left(r\right) = 303.01$. For the 99.73% one-sided control chart $E\left(r|data\right) = 555.174$ and $Median\left(r\right) = 294.31$.

See Figures 10.1, 10.2, 10.3 and 10.4 for $m = 10$ future observations.

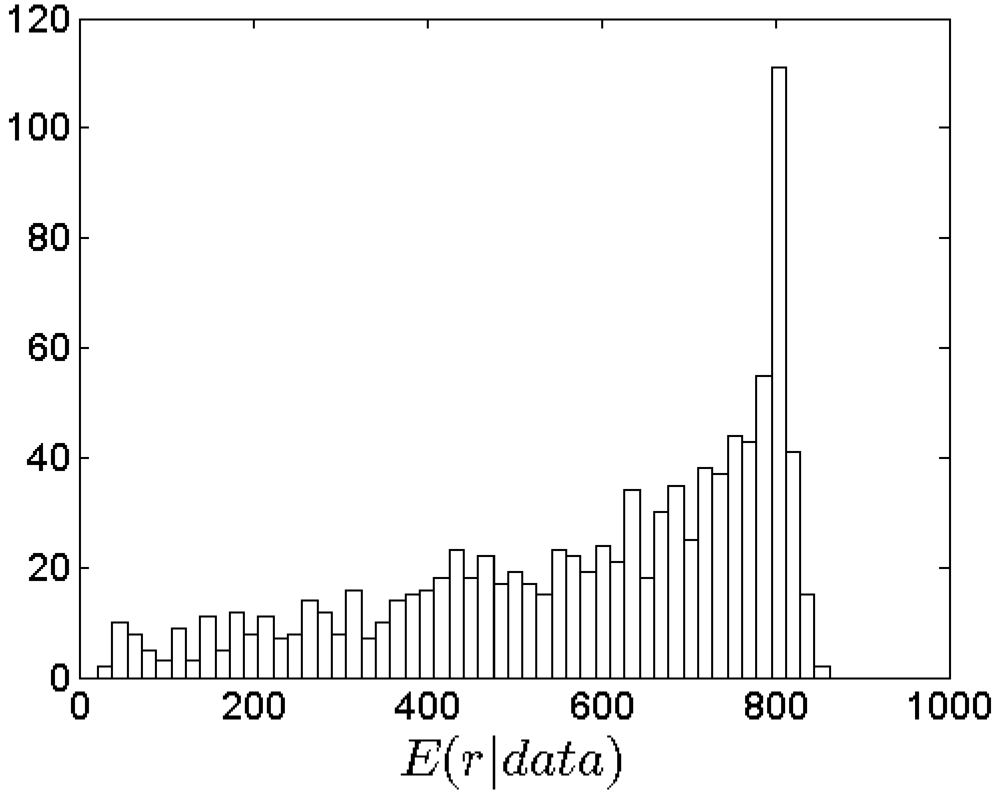
Define $\tilde{\psi}\left(\mu,\sigma^2\right) = \frac{1}{l}\sum_{i=1}^{l}\psi\left(\mu_i,\sigma_i^2\right)$. From Menzefricke (2002) it follows that if $l \to \infty$ then $\tilde{\psi} \to \beta$ and the harmonic mean of $r = \frac{1}{\beta}$. For $\beta = 0.0027$, the harmonic mean $= (0.0027)^{-1} = 370$.

Figure 10.1: Distribution of Run-length - Two-sided Interval $\beta = 0.0027$



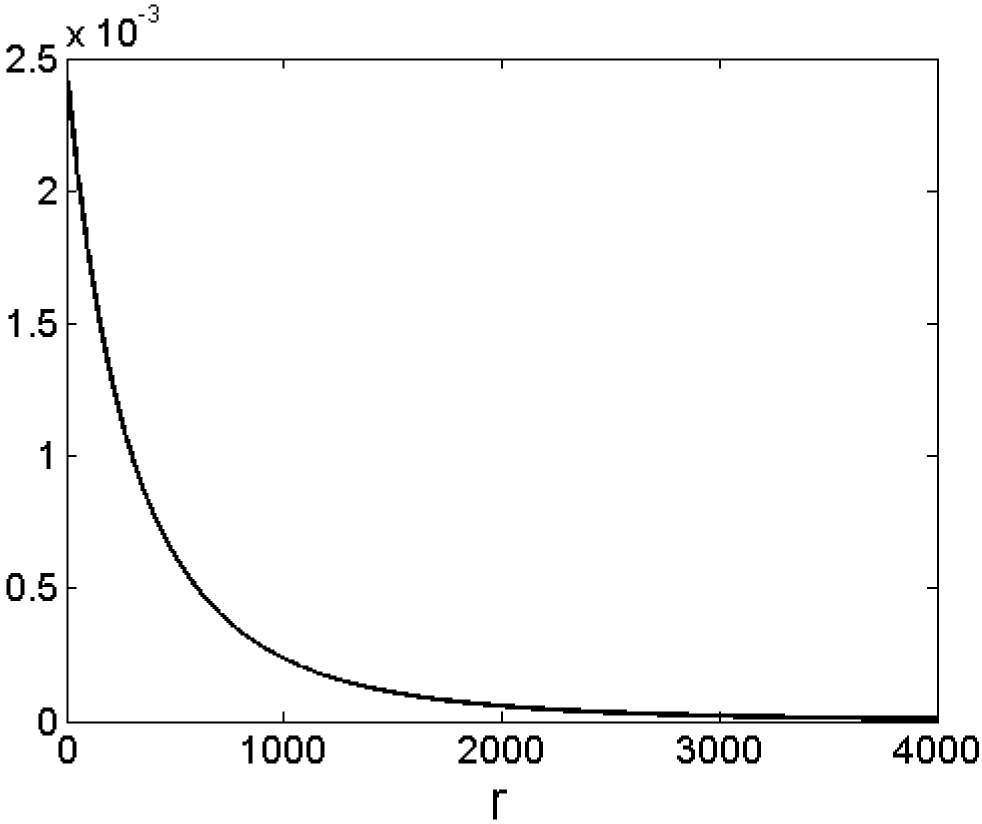
$Mean\,(r) = 482.263;\ Median\,(r) = 303.01;\ Var\,(r) = 2.8886 \times 10^5$
$95\%\ HPD\ interval = (0; 1554.5)$

Figure 10.2: Distribution of Expected Run-length - Two-sided Interval $\beta = 0.0027$



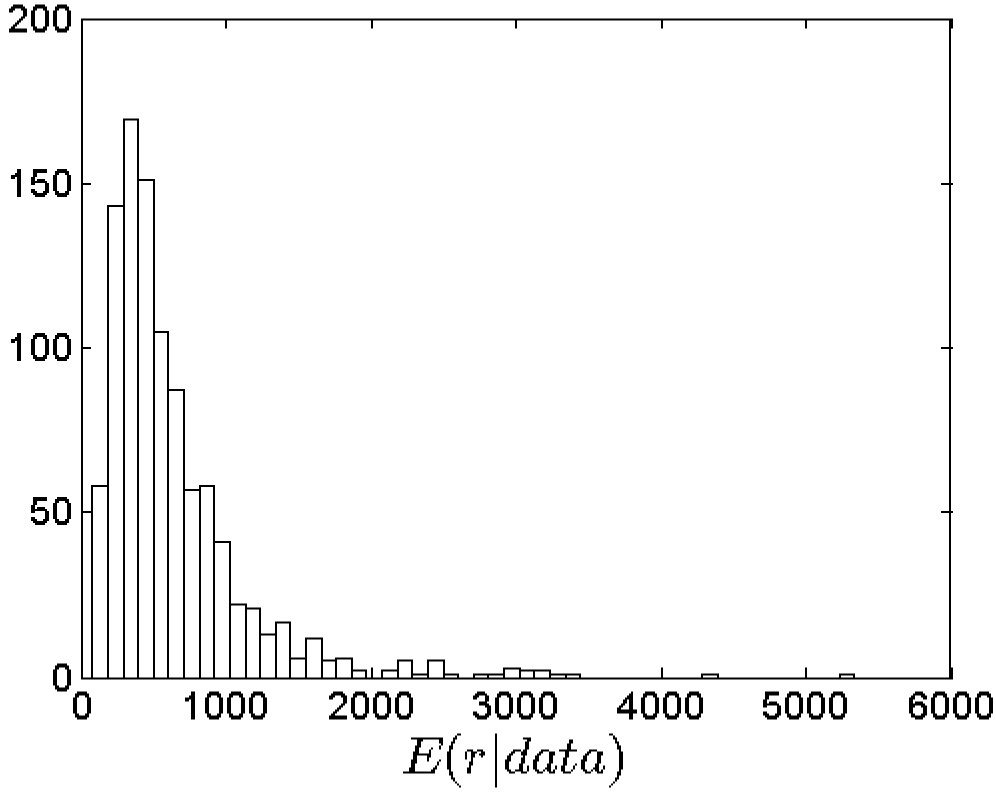
$Mean = 483.263;\ Median = 527.972,\ Var = 2.7907 \times 10^4$
$95\%\ HPD\ interval = (141.993; 689.429)$

Figure 10.3: Distribution of Run-length - One-sided Interval $\beta = 0.0027$



$Mean\,(r) = 555.174;\;\; Median\,(r) = 294.31;\;\; Var\,(r) = 6.1818 \times 10^5$
$95\%\ HPD\ interval = (0; 1971.1)$

Figure 10.4: Distribution of Expected Run-length - One-sided Interval $\beta = 0.0027$



$Mean = 563.578;\ Median = 436.528,\ Var = 2.0013 \times 10^5$
$95\%\ HPD\ interval = (0; 1384.5)$

11 Conclusion

This paper developed a Bayesian method to analyze C_{pl}, C_{pu} and C_{pk}. Multiple testing strategies have been implemented on data representing four processes from four suppliers that produce piston rings for automobile engines studied by Chou (1994). The results have been compard to the frequentist results and it was shown that Tukey's method is somewhat more conservative. Control charts were further developed using the Bayesian approach for C_{pk}.

Mathematical Appendix

Proof of Theorem 3.1

Since

$$\mu|\sigma^2, \underline{y} \sim N\left(\bar{y}, \frac{\sigma^2}{n}\right)$$

and

$$k = \frac{vS^2}{\sigma^2} \sim \chi^2_v$$

it follows that

$$t|\tilde{t}, k \sim N\left(a\sqrt{k}, \frac{1}{9n}\right)$$

where

$$a = \frac{\tilde{t}}{\sqrt{v}}.$$

Therefore

$$\begin{aligned}
p\left(t|\tilde{t}\right) &= \textstyle\int_0^\infty f\left(t|\tilde{t},k\right) f\left(k\right) dk \\
&= \tfrac{3\sqrt{n}}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)\sqrt{2\pi}} \textstyle\int_0^\infty \exp\left[-\frac{9n}{2}\left(t-a\sqrt{k}\right)^2\right] k^{\frac{v}{2}-1}\exp\left[-\frac{k}{2}\right] dk \\
&= \tfrac{3\sqrt{n}\exp\left(-\frac{9nt^2}{2}\right)}{2^{\frac{v}{2}}\sqrt{2\pi}\Gamma\left(\frac{v}{2}\right)} \textstyle\int_0^\infty k^{\frac{v}{2}-1}\left(\sum_{j=0}^\infty \frac{\left(9nta\sqrt{k}\right)^j}{j!}\exp\left[-\frac{k}{2}\left(1+9na^2\right)\right]\right) dk.
\end{aligned}$$

Since

$$\int_0^\infty k^{\frac{1}{2}(v+j)-1}\exp\left[-\frac{k}{2}\left(1+9na^2\right)\right] dk = \frac{2^{\frac{1}{2}(v+j)}\Gamma\left(\frac{v+j}{2}\right)}{\left(1+9na^2\right)^{\frac{1}{2}(v+j)}}$$

and substituting $a = \frac{\tilde{t}}{\sqrt{v}}$, the posterior distribution of $t = \frac{\mu - l}{3\sigma} = C_{pl}$ follows as

$$p\left(t|\tilde{t}\right) = \frac{3\sqrt{n}\exp\left(-\frac{9nt^2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{2\pi}} \sum_{j=0}^{\infty}\left(\frac{9nt\tilde{t}}{\sqrt{v}}\right)^j \frac{1}{j!}\frac{\Gamma\left(\frac{v+j}{2}\right)2^{\frac{1}{2}j}}{\left(1+\frac{9n}{v}\tilde{t}^2\right)^{\frac{1}{2}(v+j)}} \qquad -\infty < t < \infty.$$

Proof of Theorem 4.1

The C_{pk} index can also be written as

$$C = C_{pk} = \frac{u - l - 2\left|\mu - M\right|}{6\sigma}$$

where

$$M = \frac{u+l}{2}.$$

Since

$$\mu|\sigma^2, \underline{y} \sim N\left(\bar{y}, \frac{\sigma^2}{n}\right),$$

it follows that

$$\mu - M \sim N\left(\zeta, \frac{\sigma^2}{n}\right)$$

where

$$\zeta = \bar{y} - M.$$

Let

$$w = |\mu - M|,$$

then

$$p\left(w|\sigma^2, \underline{y}\right) = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{n}{2}\frac{(w-\zeta)^2}{\sigma^2}\right\} + \frac{n}{\sigma\sqrt{2\pi}} \exp\left(-\frac{n}{2}\frac{(w+\zeta)^2}{\sigma^2}\right)$$

(See Kotz and Johnson (1993, page 26)).

Now $C = b - \tilde{a}w$, where $\tilde{a} = \frac{1}{3\sigma}$ and $b = C_p = \frac{u-l}{6\sigma}$.

Also $w = -(C-b)\frac{1}{\tilde{a}}$ and $\left|\frac{dw}{dc}\right| = \frac{1}{\tilde{a}}$.

From this it follows that

$$p\left(C|\sigma^2, \underline{y}\right) = \frac{\sqrt{n}}{\tilde{a}\sigma\sqrt{2\pi}} \left\{\exp\left(-\frac{n}{2\tilde{a}^2\sigma^2}\left[C - b + \tilde{a}\zeta\right]^2\right) + \exp\left(-\frac{n}{2\tilde{a}^2\sigma^2}\left[C - b - \tilde{a}\zeta\right]^2\right)\right\} \quad C < \tilde{b} < \frac{S}{\sigma}$$

where $\tilde{b} = \hat{C}_p = \frac{u-l}{6s}$.

Substituting for $\tilde{a}$, b and ζ and making use of the fact that $k = \frac{vS^2}{\sigma^2} \sim \chi_v^2$ it follows that

$$p\left(C|k, \underline{y}\right) = \frac{3\sqrt{n}}{\sqrt{2\pi}} \left\{\exp\left(-\frac{9n}{2}\left[C - t^*\sqrt{\frac{k}{v}}\right]^2\right) + \exp\left(-\frac{9n}{2}\left[C - \tilde{t}\sqrt{\frac{k}{v}}\right]^2\right)\right\} \quad C < \tilde{b}\sqrt{\frac{k}{v}}.$$

Therefore

$$p\left(C|\underline{y}\right) = \frac{3\sqrt{n}}{\sqrt{2\pi}} \int_{\frac{C^2 v}{b^2}}^{\infty} \left\{\exp\left(-\frac{9n}{2}\left[C - t^*\sqrt{\frac{k}{v}}\right]^2\right) + \exp\left(-\frac{9n}{2}\left[C - \tilde{t}\sqrt{\frac{k}{v}}\right]^2\right)\right\}$$

$$\times \frac{1}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} k^{\frac{v}{2}-1} \exp\left(-\frac{k}{2}\right) dk.$$

References

Chen, K., Pearn, W., 1997. An application of non-normal process capability indices. Quality and Reliability Engineering International 13, 355–360.

Chou, Y.-M., 1994. Selecting a better supplier by testing process capability indices. Quality Engineering 6, 427–438.

Chou, Y.-M., Owen, D., 1989. On the distributions of the estimated process capability indices. Communications in Statistics - Theory and Methods 18 (12), 4549–4560.
URL `http://dx.doi.org/10.1080/03610928908830174`

Ganesh, N., 2009. Simultaneous credible intervals for small are estimation problems. Journal of Multivariate Analysis 100, 1610–1621.

Hoskins, J., Stuart, B., Taylor, J., 1988. A motorola commitment: A six sigma mandate. the motorola guide to statistical process control for continuous improvement towards six sigma quality.

Hubele, N., Shahriari, H., Cheng, C., 1991. A bivariate process capability vector. Marcel Dekker, 299–310.

Kane, V., 1986. Process capability indices. Journal of Quality Technology 18, 41–52.

Kotz, S., Johnson, N., 1993. Process Capability Indices. Chapman & Hall.

Lyth, D., Rabiej, R., 1995. Critical variables in wood manufacturing's process capability: Species, structure, and moisture content. Quality Engineering 8 (2), 275–281.

Menzefricke, U., 2002. On the evaluation of control chart limits based on predictive distributions. Communications in Statistics - Theory and Methods 31(8), 1423–1440.

Menzefricke, U., 2007. Control chart for the generalized variance based on its predictive distribution. Communications in Statistics - Theory and Methods 36(5), 1031–1038.

Menzefricke, U., 2010a. Control chart for the variance and the coefficient of variation based on their predictive distribution. Communications in Statistics - Theory and Methods 39(16), 2930–2941.

Menzefricke, U., 2010b. Multivariate exponentially weighted moving average chart for a mean based on its predictive distribution. Communications in Statistics - Theory and Methods 39(16), 2942–2960.

Niverthi, M., Dey, D., 2000. Multivariate process capability: A bayesian perspective. Communication in Statistics - Simulation and Computation 29 (2), 667–687.
URL http://dx.doi.org/10.1080/03610910008813634

Pearn, W., Wu, C., 2005. A bayesian approach for assessing process precision based on multiple samples. European Journal of Operational Research 165 (3), 685–695.

Polansky, A., 2006. Permutation methods for comparing process capabilities. Journal of Quality Technology 38 (3), 254–266.

Smit, C., Chakraborti, S., 2009. Sample capabilitiy indices: Didistribution properties and some applications. South African Statistical Journal 43, 117–125.

Zellner, A., 1971. An Introduction to Bayesian Inference in Econometrics. Wiley.