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Outlier Detection in a Random Effects Model

By

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Abstract

The most common Bayesian approach for detecting outliers is to assume that outliers are observations which have been generated by contaminating models. An alternative idea was used by Zellner (1975) and Chaloner (1994). They studied the properties of realized regression error terms. Posterior distributions for individual realized errors and for linear and quadratic combinations of them were derived. In this note the theory and results derived by Chaloner (1994) are extended. Since it is not clear to us what the frequentist properties of the Bayesian procedures of Chaloner and Zellner are (i.e. what the size of the Type I error or the power of their tests are) a Bayesian-frequentist approach is used for detecting outliers in a one-way random effects model. For illustration purposes, the Sharples (1990) contaminated data are used. It is concluded that the Bayesian frequentist approach seems to be more conservative than Chaloner's method.

Keywords: Random effects model; posterior distributions; outlying observations; Bayesian-frequentist approach; predictive distribution; control limits

1. Introduction

The most common Bayesian approach for detecting outliers is to assume that outliers are observations which have been generated by contaminating models different from the one generating the rest of the data. These contaminating models are usually considered to be either mean-shift or inflated-variance models. Previous research along these lines are given in Box and Tiao (1968), Freeman (1980), Pettit and Smith (1985), Verdinelli and Wasserman (1991) and Hoeting, Raftery and Madigan (1996).

Approaches for Bayesian checking of hierarchical models are those proposed by Dey, Gelfand, Swartz and Vlachos (1998), O'Hagan (2003), Marshall and Spiegelhalter (2003) and Bayarri and Castellanos (2007).

An alternative idea was used by Zellner (1975). He studied the properties of realized regression error terms. Posterior distributions for individual realized errors and for linear and quadratic combinations of them were derived. Zellner and Moulton (1985) on the other hand used the posterior distributions of the realized error terms to construct a residual plot. The approach used by Chaloner and Brant (1988) is an extention of the ideas applied by Zellner (1975) and Zellner and Moulton (1985). They defined an outlier to be an observation with a large realized error, generated by the model under consideration. Chaloner and Brant (1988) calculated the exact posterior probability of an observation being an outlier as well as the joint posterior probability of any two observations being outliers. Let $\varepsilon_i \sim N(0, \sigma^2)$, $i = 1, 2, \ldots, m$, and independently of each other. The $\varepsilon_1, \ldots, \varepsilon_m$ are realized errors or residuals. An outlier is defined as any observation with $|\varepsilon_i| > \tilde{k}$ for a suitable value of $\tilde{k}$. If

$$\tilde{k} = \sigma\Phi^{-1}\left\{0.5 + \frac{1}{2}\left(0.95^{\frac{1}{m}}\right)\right\} \tag{1.1}$$

then the prior probability of no outlier is 0.95. Given the data, the posterior probabilities can be calculated. According to Chaloner (1994), any observation with a posterior probability, $pr(|\varepsilon_i| > \tilde{k}|data)$ larger than the prior probability $2\Phi(-\tilde{k})$ would be a possible outlier.

In this note the theory and results derived by Chaloner (1994) for the balanced one-way random effects model will be extended. The data that will be used are the Sharples (1990) contaminated data and the possible outliers are indicated by asterisks in Table 1.1.

2. The Model and the Example

The balanced one-way random effects model is defined as

$$Y_{ij} = \theta + r_i + e_{ij} \quad (i = 1, \ldots, I,\ j = 1, \ldots, J) \tag{1.2}$$

where

$e_{ij} \sim N(0, \sigma_1^2)$; $r_i \sim N(0, \sigma_2^2)$, independently of each other.

Also

$$v_1 = I(J-1) \;;\; v_2 = I-1 \;;\; Y_{i.} = \sum_{j=1}^{J} Y_{ij} \;;$$

$$\bar{Y}_{i.} = \frac{1}{J} Y_{i.} \;;\; Y_{..} = \sum_{i=1}^{I}\sum_{j=1}^{J} Y_{ij} \;;\; \bar{Y}_{..} = \frac{1}{IJ} Y_{..} \;;$$

$$v_1 m_1 = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(Y_{ij} - \bar{Y}_{i.}\right)^2 \;;\; v_2 m_2 = J\sum_{i=1}^{I}(\bar{Y}_{i.} - \bar{Y}_{..})^2$$

where $v_1 m_1$ and $v_2 m_2$ are the within and between groups sum of squares respectively.

Example 1.1

Table 1.1: Sharples Generated Data with Possible Outliers indicated by an Asterisk

Group	**Measurements**						$\bar{Y}_{i.}$
1	24.80	26.90	26.65	30.93	33.77	**63.31***	34.39
2	23.96	28.92	28.19	26.16	21.34	29.46	26.34
3	18.30	23.67	14.47	24.45	24.89	28.95	22.46
4	**51.42***	27.97	24.76	26.67	17.58	24.29	28.78
5	34.12	46.87	**58.59***	38.11	47.59	44.67	44.99

$\bar{Y}_{..} = 31.39$

$I = 5$, $J = 6$, $v_1 = I(J-1) = 25$, $v_2 = I - 1 = 4$,

$v_1 m_1 = 2282.0893$, $v_2 m_2 = 1837.0937$

$\hat{\sigma}_1^2 = 91.2836$, $\hat{\sigma}_2^2 = 61.3316$

3. Prior and Posterior Distributions

As prior the Jeffreys' independent prior

$$p(\theta, \sigma_1^2, \sigma_2^2) \propto \sigma_1^{-2}(\sigma_1^2 + J\sigma_2^2)^{-1}$$

will be used. See for example Box and Tiao (1973, Ch.5). It can easily be shown that, given the variance components, the posterior distribution of $e_{ij} = Y_{ij} - \theta - r_i$ is normal with

$$E\left[e_{ij}\middle|\underline{Y}, \sigma_1^2, \sigma_2^2\right] = Y_{ij} - \frac{J\sigma_2^2\bar{Y}_{i.} + \sigma_1^2\bar{Y}_{..}}{\sigma_1^2 + J\sigma_2^2} \text{ and } Var\left[e_{ij}\middle|\underline{Y}, \sigma_1^2, \sigma_2^2\right] = \frac{\sigma_1^2}{IJ}\left\{\frac{\sigma_1^2 + IJ\sigma_2^2}{\sigma_1^2 + J\sigma_2^2}\right\} \quad (3.1)$$

Also, given the variance components, the posterior distribution of r_i is normal with

$$E\left[r_i\middle|\underline{Y}, \sigma_1^2, \sigma_2^2\right] = \frac{J\sigma_2^2}{\sigma_1^2 + J\sigma_2^2}(\bar{Y}_{i.} - \bar{Y}_{..}) \text{ and } Var\left[r_i\middle|\underline{Y}, \sigma_1^2, \sigma_2^2\right] = \frac{\sigma_2^2(I\sigma_1^2 + J\sigma_2^2)}{I(\sigma_1^2 + J\sigma_2^2)} \quad (3.2)$$

The posterior distribution of the variance components is given by

$$p(\sigma_1^2, \sigma_2^2|\underline{Y}) \propto (\sigma_1^2)^{-\frac{1}{2}(v_1+2)}(\sigma_1^2 + J\sigma_2^2)^{-\frac{1}{2}(v_2+2)} exp\left\{-\frac{1}{2}\left[\frac{v_1 m_1}{\sigma_1^2} + \frac{v_2 m_2}{\sigma_1^2 + J\sigma_2^2}\right]\right\}$$

$$\sigma_1^2 > 0 \;;\; \sigma_2^2 > 0 \text{ and } \sigma_1^2 + J\sigma_2^2 = \sigma_{12}^2 > \sigma_1^2. \tag{3.3}$$

The posterior distribution of the variance components as well as that of $\frac{e_{ij}}{\sigma_1}$ and $\frac{r_i}{\sigma_2}$ are given in Chaloner (1994).

Simulation of the variance components follow easily from (3.3) as follows:

Since $\frac{v_1 m_1}{\sigma_1^2} \sim \chi_{v_1}^2$ and $\frac{v_2 m_2}{\sigma_{12}^2} \sim \chi_{v_2}^2$, σ_1^2, σ_{12}^2 and $\sigma_2^2 = \frac{\sigma_{12}^2 - \sigma_1^2}{J}$ can easily be simulated. If a negative value of σ_2^2 is obtained, disregard this value as well as the corresponding σ_1^2 value. It is our opinion that this is the best method to simulate the variance components if the percentage of negative σ_2^2 values are zero or very small. If the number of negative σ_2^2 values is less than one percent, this procedure will give correct frequentist coverage. For the data in Table 1.1 the percentage of negative simulated σ_2^2 values was only 0.41%. If on the other hand the between group variance, σ_2^2 is small so that the number of negative simulated σ_2^2 values are large, then $p(\theta, \sigma_1^2, \sigma_2^2) \propto \sigma_1^{-2}\sigma_2^{-1}$ might be a better prior to use. It is well known that the prior $p(\theta, \sigma_1^2, \sigma_2^2) \propto \sigma_1^{-2}\sigma_2^{-2}$ will give improper posterior distributions. The simulated values can then be substituted in the conditional normal distributions (for example equations (3.1) and (3.2)). Repeat the above steps until $\tilde{l}$ permissible values are obtained. For the Sharples data $\tilde{l}$ will be taken as 100 000. By averaging the $\tilde{l}$ conditional distributions (Rao-Blackwell method), the unconditional posterior distributions can be obtained.

4. <u>The Bayesian Method of Chaloner</u>

In this section the results derived by Chaloner (1994) will be illustrated in a different form – as a ratio of the prior probability to the posterior probability. No new contributions to the existing theory are therefore made in this section. Chaloner (1994) claimed that any observation with a posterior probability larger than the prior probability would be a possible outlier.

She defined two sets of residuals:

The within-group residuals $\frac{e_{ij}}{\sigma_1}$ and the $\frac{r_i}{\sigma_2}$ between-group residuals. The within-group residuals measure how far Y_{ij} is from its mean and the between-group residuals measure how far r_i is from zero.

The unconditional posterior distributions of these residuals can easily be obtained by simulating the variance components from equation (3.3). Using equation (1.1) for a sample of size 30 gives $\tilde{k} = 3.14$. The prior probability of an observation being an outlier is therefore 0.0017. The posterior probabilities that observations Y_{16}, Y_{41} and Y_{53} are outliers are

(a) $P\left(\left|\frac{e_{16}}{\sigma_1}\right| > \tilde{k}|\underline{Y}\right) = 0.43810$, (b) $P\left(\left|\frac{e_{41}}{\sigma_1}\right| > \tilde{k}|\underline{Y}\right) = 0.04963$ and

(c) $P\left(\left|\frac{e_{53}}{\sigma_1}\right| > \tilde{k}|\underline{Y}\right) = 0.00059$. The corresponding ratios are

(i) $\widetilde{BF}_{16} = \frac{0.43810}{0.0017} = 257.71$, (ii) $\widetilde{BF}_{41} = \frac{0.04963}{0.0017} = 29.19$ and (iii) $\widetilde{BF}_{53} = \frac{0.00059}{0.0017} = 0.35$.

Y_{16} and Y_{41} are therefore possible outliers. $\widetilde{BF}$ is not a Bayes factor in the true sense of the word. It is as mentioned only a ratio and another way to indicate the magnitude of the difference between the prior and posterior probabilities.

Since Jeffreys' prior is not proper, Bayes factors cannot be calculated in the usual way. Partial Bayes factors (intrinsic or fractional) were also not calculated because this was not the purpose of the study.

Similar results can be obtained for the between-group residuals. Since there are only five groups, it follows from equation (1.1) that for $m = 5$, $\tilde{k} = 2.57$ the prior probability that any one of the groups will be outlying is 0.0102. The posterior probabilities are (a) $P\left(\left|\frac{r_1}{\sigma_2}\right| > \tilde{k}|\underline{Y}\right) = 0.000431$, (b) $P\left(\left|\frac{r_3}{\sigma_2}\right| > \tilde{k}|\underline{Y}\right) = 0.00614$ and

(c) $P\left(\left|\frac{r_5}{\sigma_2}\right| > \tilde{k}|\underline{Y}\right) = 0.03393$. It therefore seems that Group 5 is a possible outlier because its posterior probability is larger than the prior probability of 0.0102. The corresponding ratios are

(i) $\widetilde{BF}_1 = \frac{0.000431}{0.0102} = 0.042$, (ii) $\widetilde{BF}_2 = \frac{0.00614}{0.0102} = 0.60$ and (iii) $\widetilde{BF}_3 = \frac{0.03393}{0.0102} = 3.33$.

Chaloner (1994) also mentioned that if the analysis is repeated after deleting the three outlying observations, Y_{16}, Y_{41} and Y_{53}, the outlying nature of Group 5 is even more apparent as the approximate posterior probability that Group 5 is outlying is 0.0429.

In this case $\widetilde{BF}_5^* = \frac{0.0429}{0.0102} = 4.25$. A comparison with Jeffreys' scale of evidence for Bayes factors (although it is not a fair comparison because $\widetilde{BF}_5^*$ is not a Bayes factor in the true sense of the word) shows that there is evidence that Group 5 is outlying because the value is between 3 and 10.

In our opinion, Group 5 is not an outlying group. It's t-value given in Table 5.2 is not even significant at the 5% level. This is also apparent from Figures 5.1 and 5.3.

5. A Bayesian-Frequentist Approach for Outlier Detection

5.1. The Known-Variance Case

Since it is not clear to us what the frequentist properties of the Bayesian procedures of Chaloner and Zellner are (i.e. what the size of the Type I error or the power of their tests are) a Bayesian-frequentist approach will be used for detecting outliers in a one-way random effects model. According to Bayarri and Berger (2004), statisticians should readily use both Bayesians and frequentist ideas. Objective Bayesians and frequentist methods often give similar results for normal linear models. See for example the results in Tables 5.1 and 5.2. Reid and Cox (2014) on the other hand mentioned that "A hybrid method of inference that uses Bayesian reasoning with impersonal priors, if the results are well calibrated in the frequency sense, may be ideal, but to date the construction of these priors is elusive." It is a very desirable situation if the resulting Bayesian procedure will also have good frequentist properties.

From equation (3.2) it follows that the standardised random effect $r_i^* = \frac{r_i}{\sqrt{Var[r_i|\sigma_1^2, \sigma_2^2, data]}}$

is also normally distributed with $E[r_i^*|\sigma_1^2, \sigma_2^2, data] = \frac{J\sigma_2\sqrt{I}}{\sqrt{(\sigma_1^2+J\sigma_2^2)(I\sigma_1^2+J\sigma_2^2)}}(\bar{Y}_{i.} - \bar{Y}_{..})$ and $Var[r_i^*|\sigma_1^2, \sigma_2^2, data] = 1$.

In Table 5.1 the Bayesian results for the random effects are illustrated and in Table 5.2 the SAS printout for the Sharples data are given. Since the true parameter values are not known, the point estimates $\hat{\sigma}_1^2 = 91.2836$ and $\hat{\sigma}_2^2 = 61.3316$ are used in the formulae.

Table 5.1: Solution for Random Effects - Bayes Procedure - Variances Known

Group	$E[r_i\|\sigma_1^2, \sigma_2^2, data]$	$\sqrt{Var[r_i\|\sigma_1^2, \sigma_2^2, data]}$	$E[r_i^*\|\sigma_1^2, \sigma_2^2, data]$
1	2.4048	4.6924	0.5125
2	-4.0492	4.6924	-0.8629
3	-7.1607	4.6924	-1.5260
4	-2.0915	4.6924	-0.4457
5	10.8967	4.6924	2.3222

Table 5.2: Solution for Random Effects – SAS – Satterthwaite Procedure

Group	Estimate	Std Err Pred	DF	t Value	Pr>|t|
1	2.4048	4.6924	6.11	0.51	0.6263
2	-4.0492	4.6924	6.11	-0.86	0.4207
3	-7.1607	4.6924	6.11	-1.53	0.1769
4	-2.0915	4.6924	6.11	-0.45	0.6711
5	10.8966	4.6924	6.11	2.32	0.0585

A comparison between the two tables show that $\sqrt{Var[r_i|\sigma_1^2,\sigma_2^2,data]}$ is equal to the standard error of a predictor (Std Err Pred) and the expected value of the standardized random effect $E[r_i^*|\sigma_1^2,\sigma_2^2,data]$ is equal to the t Value in the SAS Printout.

As mentioned the expected values of the standardised residuals will be used as measures for detecting possible outliers. If in a certain data set

$|E[r_i^*|\sigma_1^2,\sigma_2^2,data]| > \tilde{k}$ then r_i will be considered an outlier. The values of $\tilde{k}$ will be obtained from the predictive distribution of $E[r_i^*|\sigma_1^2,\sigma_2^2]$.

5.2. The Predictive Distribution of $E\left[r_i^*|\sigma_1^2,\sigma_2^2\right] = Y_i^*$

Let $Y_i^* = \frac{J\sigma_2\sqrt{I}(\bar{Y}_{i.}-\bar{Y}_{..})}{\sqrt{(\sigma_1^2+J\sigma_2^2)(I\sigma_1^2+J\sigma_2^2)}}$

where $\bar{Y}_{i.}$ $(i = 1, \ldots, 5)$ and $\bar{Y}_{..}$ are considered to be random variables. The predictive distribution of Y_i^* will tell us what possible values $E[r_i^*|\sigma_1^2,\sigma_2^2]$ might take on in future experiments. The following theorem can now be proved:

<u>Theorem 5.1</u>

If H$_0$ is true (i.e if the model is correct) then $\underline{Y}^* = [Y_1^* \quad Y_2^* \quad \ldots \quad Y_I^*]'$ given the variance components is multivariate normally distributed with mean $E[Y_i^*] = 0$ and variance $Var[Y_i^*] = \frac{J(I-1)\sigma_2^2}{(I\sigma_1^2+J\sigma_2^2)}$ $(i = 1, \ldots, I)$. The correlation coefficient between Y_l^* and $Y_m^* = \rho_{l,m} = \frac{-1}{I-1}$, $l = 1, \ldots, I,\ m = 1, \ldots, I, l \neq m.$

The proof is given in Appendix A.

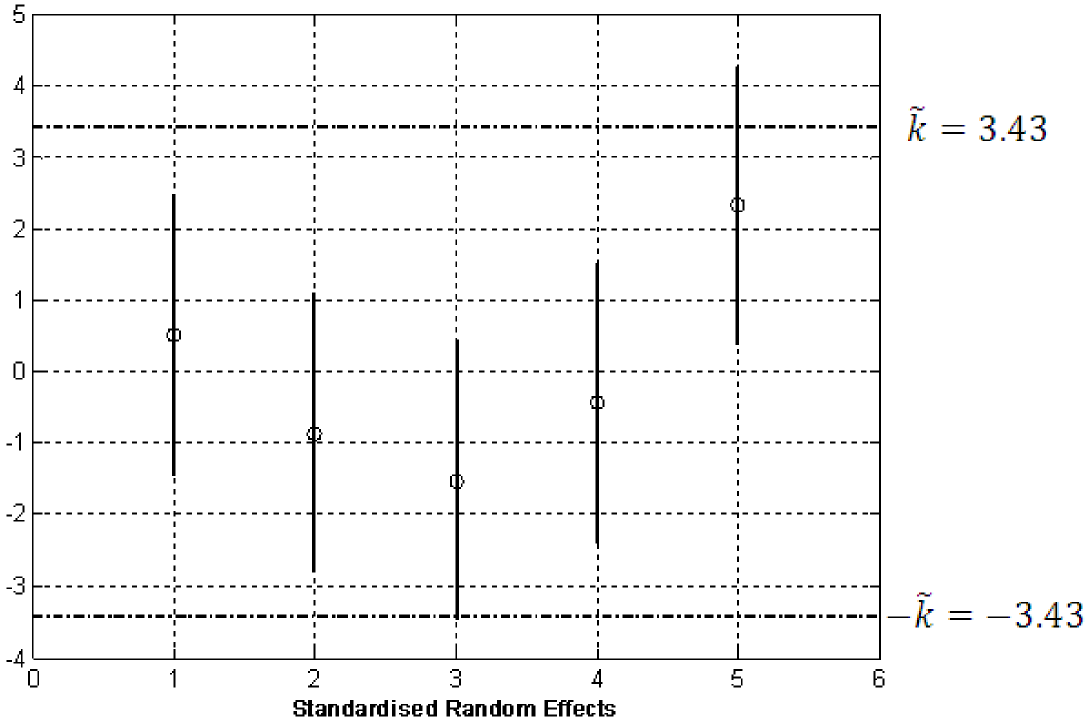
Three cases will be considered in the section to obtain the limit $\tilde{k}$. In the first case the point estimates will be substituted for the parameter values.

From Theorem 5.1 it follows that if $I = 5$,

$$\tilde{k} = 2.57\sqrt{Var[Y_i^*]} = 2.57\sqrt{\frac{(I-1)J\sigma_2^2}{(I\sigma_1^2+J\sigma_2^2)}} = 2.57\sqrt{\frac{(4)(6)(61.3316)}{5(91.2836)+6(61.3316)}} = 3.4341.$$

This means that in 95% of future experiments all the $E[r_i^*|\sigma_1^2, \sigma_2^2, data]$, $i = 1, \ldots, 5$ will fall between -3.4341 and 3.4341. In Figure 5.1 the control limits for the Sharples data are illustrated.

Figure 5.1: Means and 95% Intervals for r_i^*, $i = 1, \ldots, 5$



According to our method, Group 5 is not an outlying group, because $P(|r_5^*| > 3.4341) = 0.1441$. The mean of Group 5 is 2.32 which is smaller than 3.43.

5.3. Unknown Variances

In the second case, the unknown variance case $\tilde{k}$ can be obtained by simulation. It was mentioned in Section 3 that:

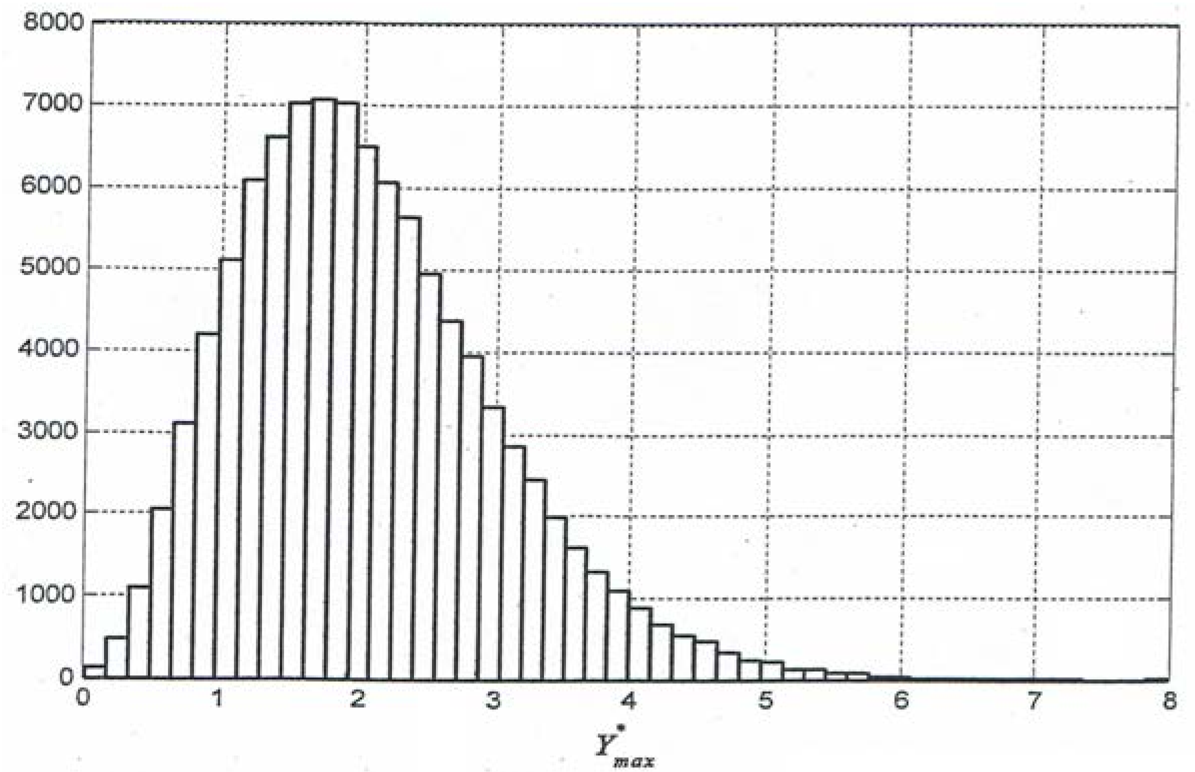
(i) Since $\frac{v_1 m_1}{\sigma_1^2} \sim \chi_{v_1}^2$ and $\frac{v_2 m_2}{\sigma_{12}^2} \sim \chi_{v_2}^2$, σ_1^2, σ_{12}^2 and $\sigma_2^2 = \frac{\sigma_{12}^2 - \sigma_1^2}{J}$ can easily be simulated.

(ii) For each pair of simulated variance components (σ_1^2, σ_2^2) draw $\underline{Y}^*$ from the multivariate normal distribution given in Theorem 5.1. Since $\underline{Y}^*$ is a singular normal distribution, only $I - 1$ random variables can be drawn. For the Sharples data draw $Y_1^*, Y_2^*, Y_3^*, Y_4^*$ and calculate $Y_5^* = 0 - \sum_{l=1}^{4} Y_l^*$.

(iii) Calculate $Y_{max}^* = \max(|Y_i^*|; i = 1,2, \ldots, 5)$.

(iv) Repeat steps (i) – (iii) 100 000 times and draw a histogram of Y_{max}^*.

Figure 5.2: Histogram of Y^*_{max}



$\boldsymbol{Y^{*0.95}_{max} = 3.8383 = \tilde{k}}$ **(100 000 simulations)**

The control limits as well as the means and 95% Bayesian confidence intervals for r_i^* are given in Figure 5.3, and in Figure 5.4 the unconditional posterior distributions of r_i^* are displayed. The unconditional posterior distributions are obtained by averaging the conditional posterior distributions, i.e. the Rao-Blackwell method.

Figure 5.3: Means and 95% Intervals for r_i^*, $i = 1, \dots, 5$, Unknown Variances

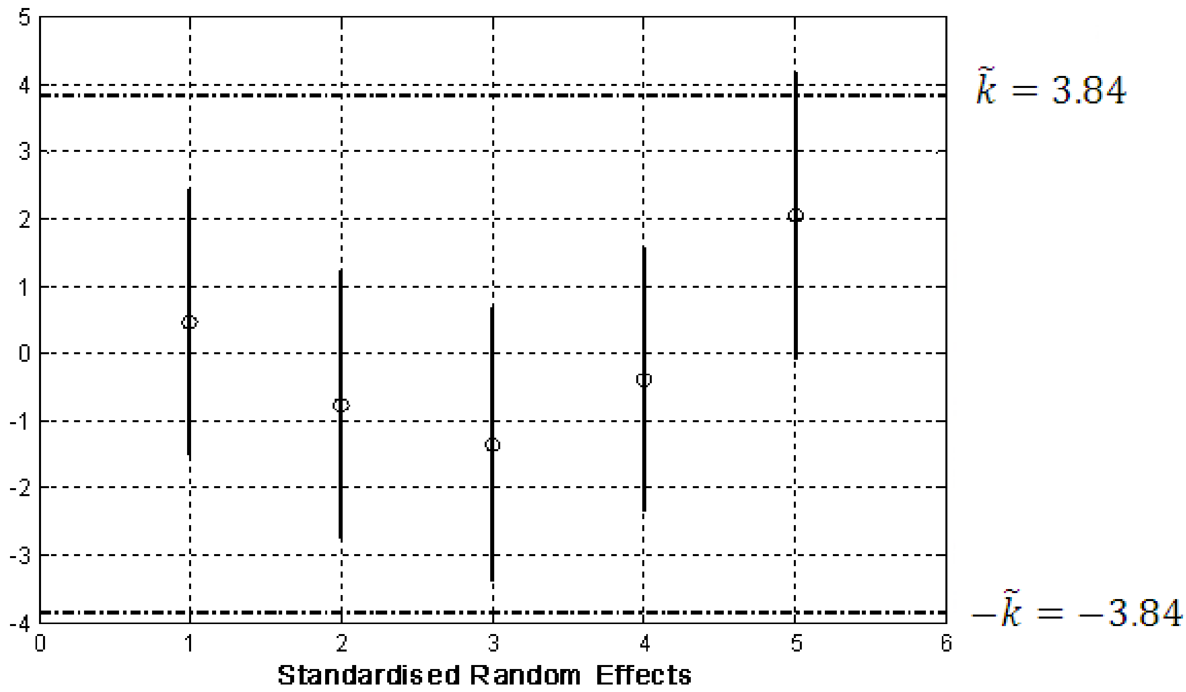
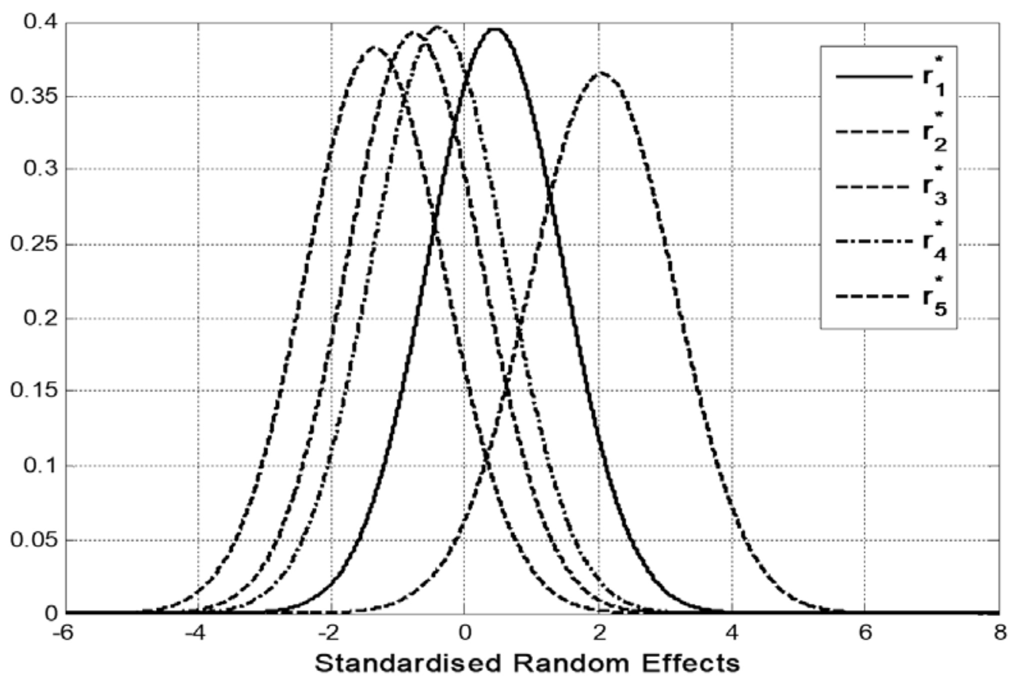


Figure 5.4: Unconditional Posteriors of r_i^*, $i = 1, \dots, 5$



In Table 5.3 the means, variances and 95% intervals for r_i^* are given, and in the last column the outlying probabilities of the five groups are given.

Table 5.3: Probabilities that Groups are outlying

r_i^*	$Mean(r_i^*)$	$Var(r_i^*)$	95% Interval	$P(Outlier)$
1	0.4507	1.0091	-1.518 – 2.420	0.0004
2	-0.7589	1.0257	-2.744 – 1.266	0.0012
3	1.3421	1.0804	-3.379 – 0.695	0.0083
4	-0.3920	1.0069	-2.359 – 1.575	0.0003
5	2.0424	1.1862	-0.092 – 4.177	0.0500

Inspection of Table 5.3 shows that the probability that Group 5 is outlying is now 0.0500 which is smaller than 0.1441, the probability for the known variance case. The estimation of the variance components using Monte Carlo simulation leads to more uncertainty and that is the reason for the smaller probability. According to the Bayesian-frequentist procedure there is no reason to believe that any one of the groups are outlying.

A possible criticism of the Bayesian-Frequentist procedure so far could have been the apparent double use of the data. The same data are used for obtaining the posterior distribution of the r_i^* and the predictive distribution of the Y_i^*. As mentioned by Bayarri and Castellanos (2007): "This can result in severe conservatism incapable of detecting clearly inappropriate models." See also Bayarri and Berger (2000) and Bayarri and Morales (2003).

For the third method, all the data were used to calculate the posterior distribution of the standardised random effects, but for predictive purposes, Y_{16}, Y_{41} and Group 5 were deleted from the dataset. For the unbalanced random effects model the Monte Carlo simulation procedure is somewhat more complicated and will be discussed in the next section. It was however found that $\tilde{k} = 3.44$. The value given in Figure 5.1 where the point estimates are substituted for the variance components is $\tilde{k} = 3.43$. These two values are for all practical purposes the same. It can therefore be concluded that group 5 is not an outlying group.

6. Outliers in the Case of Individual Observations

From equation (3.1) it is clear that the standardised residual $e_{ij}^* = \frac{e_{ij}}{\sqrt{Var\left[e_{ij}\middle|\underline{Y}, \sigma_1^2, \sigma_2^2\right]}}$ is normally distributed with mean $E\left[e_{ij}^*\middle|\underline{Y}, \sigma_1^2, \sigma_2^2\right] = \frac{Y_{ij} - \frac{J\sigma_2^2}{\sigma_1^2 + J\sigma_2^2}\bar{Y}_{i.} - \frac{\sigma_1^2}{\sigma_1^2 + J\sigma_2^2}\bar{Y}_{..}}{\sqrt{\frac{\sigma_1^2}{\sigma_1^2 + J\sigma_2^2}\left(\sigma_2^2 + \frac{\sigma_1^2}{IJ}\right)}}$

and variance $Var\left[e_{ij}^*\middle|\underline{Y}, \sigma_1^2, \sigma_2^2\right] = 1, i = 1, \dots, I,\ j = 1, \dots, J.$

If for a certain data set $\left|E\left[e_{ij}^*\middle|\sigma_1^2, \sigma_2^2, \underline{Y}\right]\right| > k^*$ then e_{ij} will be considered an outlier. As before the value of k^* will be obtained from the predictive distribution of $E\left[e_{ij}^*\middle|\sigma_1^2, \sigma_2^2\right]$.

6.1. The Predictive Distribution of $E\left[e_{ij}^{*}\middle|\sigma_1^2,\sigma_2^2\right] = Y_{ij}^{*}$

Let $Y_{ij}^{*} = \dfrac{Y_{ij} - \frac{J\sigma_2^2}{\sigma_1^2 + J\sigma_2^2}\overline{Y}_{i.} - \frac{\sigma_1^2}{\sigma_1^2 + J\sigma_2^2}\overline{Y}_{..}}{\sqrt{\frac{\sigma_1^2}{\sigma_1^2 + J\sigma_2^2}\left(\sigma_2^2 + \frac{\sigma_1^2}{IJ}\right)}}$

where Y_{ij}, $\overline{Y}_{i.}$ and $\overline{Y}_{..}$ are considered to be future observations, i.e. random variables. The predictive distribution of Y_{ij}^{*} $(i = 1, \dots, I,\ j = 1, \dots, J)$ will be an indication of what possible values $E\left[e_{ij}^{*}\middle|\sigma_1^2,\sigma_2^2\right]$ might take on in future experiments. The following Theorem can now be proved.

Theorem 6.1

If the data are generated by the model given in equation (1.2) then $\underline{Y}^{*} = [Y_{11}^{*} \quad Y_{12}^{*} \quad \cdots \quad Y_{IJ}^{*}]'$ will be multivariate normally distributed with mean

$E\left[Y_{ij}^{*}\middle|\sigma_1^2,\sigma_2^2\right] = 0 \quad (i = 1, \dots, I,\ j = 1, \dots, J)$

$Var\left[Y_{ij}^{*}\middle|\sigma_1^2,\sigma_2^2\right] = \frac{1}{\sigma_1^2 + IJ\sigma_2^2}\{I(J-1)(\sigma_1^2 + J\sigma_2^2) + \sigma_1^2(I-1)\}$

$Cov\left[Y_{ij}^{*}, Y_{lm}^{*}\middle|\sigma_1^2,\sigma_2^2\right] = \frac{-\sigma_1^2}{\sigma_1^2 + IJ\sigma_2^2}$

$Cov\left[Y_{ij}^{*}, Y_{lj}^{*}\middle|\sigma_1^2,\sigma_2^2\right] = \frac{-\sigma_1^2}{\sigma_1^2 + IJ\sigma_2^2}$

and

$Cov\left[Y_{ij}^{*}, Y_{im}^{*}\middle|\sigma_1^2,\sigma_2^2\right] = -1$

The proof is given in Appendix B.

To avoid the double use of the data in the simulation procedure for obtaining the predictive distribution, the possible outliers Y_{16}, Y_{41} and Y_{53} are deleted from the data set. Group 5 will however not be deleted because it was shown in Section 5 that it is not an outlying group. From this it follows that $n_1 = 5, n_2 = 6, n_3 = 6, n_4 = 5$ and $n_5 = 5$. Since the sample sizes are now unequal, the prior $p(\theta, \sigma_1^2, \sigma_2^2) \propto \sigma_1^{-2}(\sigma_1^2 + J\sigma_2^2)^{-1}$ cannot be used for predictive purposes. To simplify the simulation procedure, the probability-matching prior and reference priors $\mathrm{P}_{\mathrm{R1}}(\theta, \gamma, \sigma_1^2)$ and $\mathrm{P}_{\mathrm{R2}}(\theta, \gamma, \sigma_1^2)$ are preferred. If the parameter $\gamma = \frac{\sigma_2^2}{\sigma_1^2}$ is used instead of σ_2^2, the posterior distribution can be expressed in hierarchical form. γ is first simulated and then σ_1^2. σ_2^2 follows from the product of γ and σ_1^2. In other words, ordinary Monte Carlo simulation can be used and no Gibbs sampling is necessary. These priors often lead to procedures with good frequentist properties. An in-depth discussion of the nature and merits of the reference and probability-matching priors lie outside the scope of this article, but the interested reader should consult Berger and Bernardo (1992(a) and 1992(b)) as well as Datta and Ghosh (1995).

The Probability-matching prior for the parameters $(\theta, \gamma, \sigma_1^2)$ is given by:

$$\mathrm{P_{R1}}(\theta, \gamma, \sigma_1^2) \propto \sigma_1^{-2} \left\{ \sum_{i=1}^{I} \frac{n_i^2}{(1+\gamma n_i)^2} - \frac{1}{n} \left(\sum_{i=1}^{I} \frac{n_i}{1+\gamma n_i} \right)^2 \right\}^{\frac{1}{2}} = \sigma_1^{-2} \mathrm{P_{R1}}(\gamma) \tag{6.1}$$

which is also a Reference prior for the parameter groupings $(\theta, \gamma, \sigma_1^2)$, $(\gamma, \theta, \sigma_1^2)$ and $(\gamma, \sigma_1^2, \theta)$.

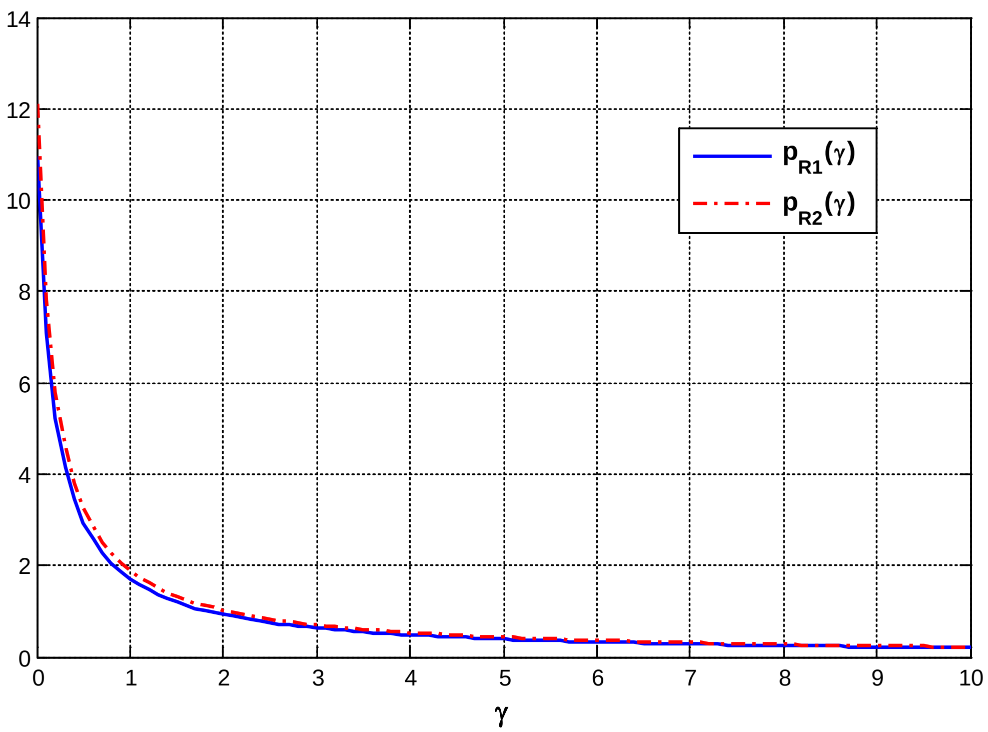
The Reference prior for the parameter groupings $(\theta, \sigma_1^2, \gamma)$, $(\sigma_1^2, \gamma, \theta)$ and $(\sigma_1^2, \theta, \gamma)$ is given by:

$$\mathrm{P_{R2}}(\theta, \gamma, \sigma_1^2) \propto \sigma_1^{-2} \left\{ \sum_{i=1}^{I} \frac{n_i^2}{(1+\gamma n_i)^2} \right\}^{\frac{1}{2}} = \sigma_1^{-2} \mathrm{P_{R2}}(\gamma) \tag{6.2}$$

where $\sum_{i=1}^{I} n_i = n$

The similarity of the priors $\mathrm{P_{R1}}(\gamma)$ and $\mathrm{P_{R2}}(\gamma)$ are illustrated in Figure 6.1.

Figure 6.1: Prior Distributions of $\gamma = \frac{\sigma_2^2}{\sigma_1^2}$



For further details about the derivations of equations (6.1) and (6.2) see for example van der Merwe, Pretorius and Meyer (2006); van der Merwe and Bekker (2006) and Harvey (2012).

If $n_1 = n_2 = \cdots = n_I = J$ both these priors simplify to Jeffreys' independent prior $\mathrm{P}(\theta, \sigma_1^2, \sigma_2^2) \propto \sigma_1^{-2}(\sigma_1^2 + J\sigma_2^2)^{-1}$.

From equations (6.1) and (6.2) it follows that the posterior distributions $\mathrm{P}_{\mathrm{R1}}(\gamma|\underline{Y})$ and $\mathrm{P}_{\mathrm{R2}}(\gamma|\underline{Y})$ are given by:

$$\mathrm{P}_{\mathrm{R1}}(\gamma|\underline{Y}) = c_1 \mathrm{P}_{\mathrm{R1}}(\gamma) \prod_{i=1}^{I} \left(\frac{1}{1+\gamma n_i}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{I} \frac{n_i}{1+\gamma n_i}\right)^{-\frac{1}{2}} \left[v_1 m_1 + \sum_{i=1}^{I} \frac{n_i(\overline{Y}_{i.} - \hat{\theta})^2}{1+\gamma n_i}\right]^{-\frac{1}{2}(n-1)} \qquad (6.3)$$

and

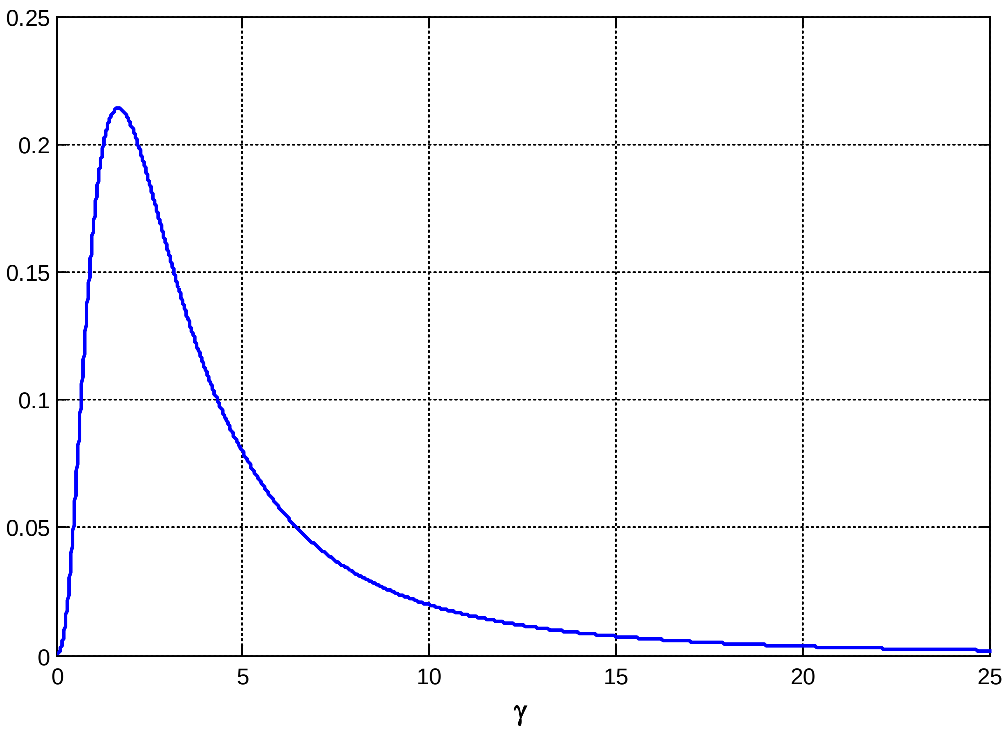
$$\mathrm{P}_{\mathrm{R2}}(\gamma|\underline{Y}) = c_2 \mathrm{P}_{\mathrm{R2}}(\gamma) \prod_{i=1}^{I} \left(\frac{1}{1+\gamma n_i}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{I} \frac{n_i}{1+\gamma n_i}\right)^{-\frac{1}{2}} \left[v_1 m_1 + \sum_{i=1}^{I} \frac{n_i(\overline{Y}_{i.} - \hat{\theta})^2}{1+\gamma n_i}\right]^{-\frac{1}{2}(n-1)} \qquad (6.4)$$

$\mathrm{v}_1 = n - I$, $\mathrm{v}_1 m_1 = \sum_{i=1}^{I} \sum_{j=1}^{n_i} (Y_{ij} - \overline{Y}_{i.})^2$, and $\hat{\theta} = \dfrac{\sum_{i=1}^{I} \overline{Y}_{i.} \frac{n_i}{1+\gamma n_i}}{\sum_{i=1}^{I} \frac{n_i}{1+\gamma n_i}}$

c_1 and c_2 are the normalising constants.

In Figure 6.2, the two posterior distributions $\mathrm{P}_{\mathrm{R1}}(\gamma|\underline{Y})$ and $\mathrm{P}_{\mathrm{R2}}(\gamma|\underline{Y})$ are displayed.

Figure 6.2: Posterior Distribution of $\mathrm{P}_{\mathrm{R1}}(\gamma|\underline{Y})$ and $\mathrm{P}_{\mathrm{R2}}(\gamma|\underline{Y})$



$Mean(\gamma) = 5.0846$, $Median(\gamma) = 3.318$, $Mode(\gamma) = 1.67$, $Var(\gamma) = 28.404$,

$c_1 = 1.329277468 \times 10^{-38}$ and $c_2 = 1.472594966 \times 10^{-38}$

The following theorem can also be proved.

Theorem 6.2

$\mathrm{P}_{\mathrm{R}1}(\gamma|\underline{Y})$ and $\mathrm{P}_{\mathrm{R}2}(\gamma|\underline{Y})$ are both proper posterior distributions.

The proof is given in Appendix C

A comparison of the normalising constants c_1 and c_2 shows that the two posterior distributions are not exactly the same. Further calculations to indicate the very small differences in the two posteriors are given in Appendix D.

However, since the two posterior distributions are for all practical purposes the same, only $\mathrm{P}_{\mathrm{R}1}(\gamma|\underline{Y})$ will be used in the Monte Carlo simulation procedure.

The joint posterior distribution can be written as

$$\mathrm{p}(\theta, \sigma_1^2, \gamma|\underline{Y}) = \mathrm{p}(\theta|\sigma_1^2, \gamma, \underline{Y})\mathrm{p}(\sigma_1^2|\gamma, \underline{Y})\mathrm{P}_{R1}(\gamma|\underline{Y})$$

and the simulation method to obtain k^* is as follows:

(i) By using a rejection method simulate γ from $\mathrm{P}_{R1}(\gamma|\underline{Y})$.

(ii) Given γ, σ_1^2 has an Inverse Gamma distribution and a simulated value of σ_1^2 can be obtained from the equation:

$$\sigma_1^2 = \frac{1}{\chi_{n-1}^2}\left[\mathrm{v}_1 m_1 + \sum_{i=1}^{I} \frac{n_i(\overline{Y}_{i.} - \hat{\theta})^2}{1 + \gamma n_i}\right]$$

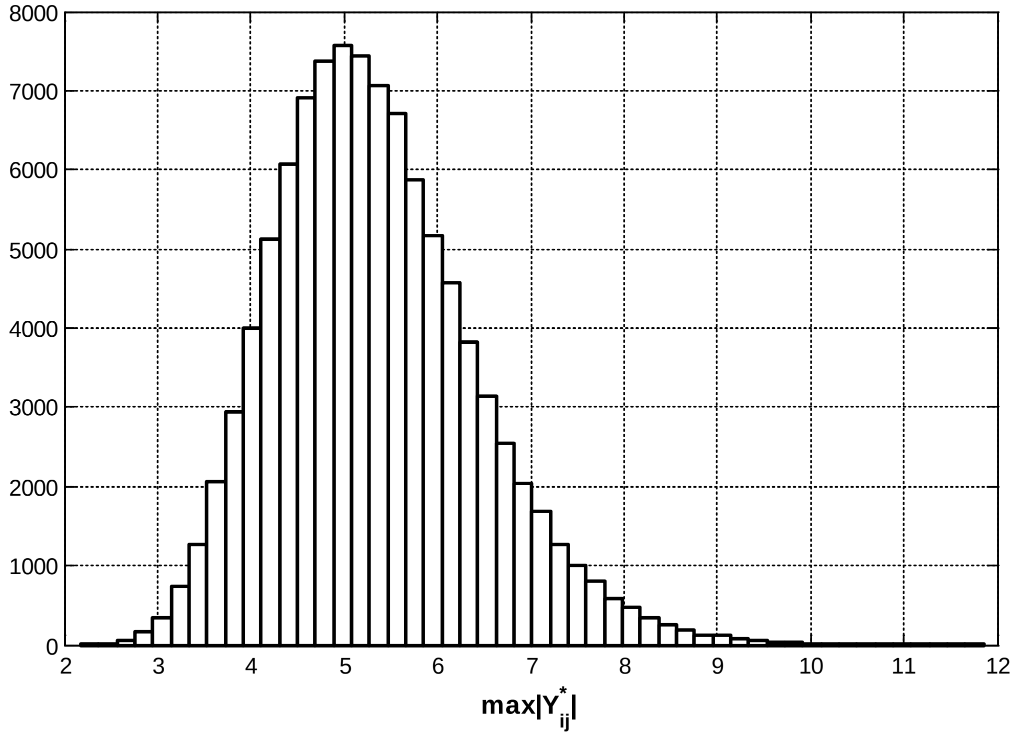
(iii) From (i) and (ii) it follows that $\sigma_2^2 = \sigma_1^2\gamma$.

(iv) For each pair of simulated variance components (σ_1^2, σ_2^2) draw $\underline{\underline{Y}}^* = [Y_{11}^* \quad Y_{12}^* \quad \dots \quad Y_{IJ}^*]'$ from the multivariate normal distribution defined in Theorem 6.1

(v) Calculate the maximum of the absolute values that have been drawn and call it $\max|Y_{ij}^*| \quad i = 1, \dots, I,\ j = 1, \dots, J$.

(vi) Repeat steps (i) – (v) 100 000 times and draw a histogram of the simulated $\max|Y_{ij}^*|$ values

Figure 6.3: Histogram of $\max\left|Y_{ij}^*\right|$



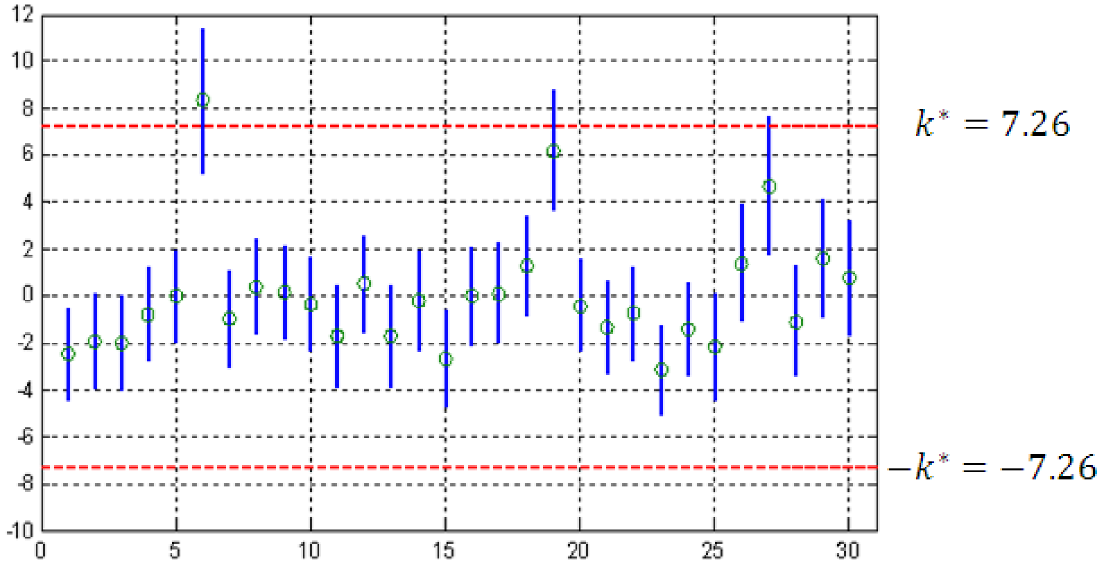
$$\max\left|Y_{ij}^*\right|^{0.95} = 7.2584 = k^*$$

$$\mathbf{Mean} = 5.3196 \;, \qquad \mathbf{Median} = 5.2119 \;, \qquad \mathbf{Mode} = 5 \;, \qquad \mathbf{Variance} = 1.1506$$

If the prior $\mathrm{p}(\theta, \sigma_1^2, \sigma_2^2) \propto \sigma_1^{-2}\sigma_2^{-1}$ is used (see for example Bayarri and Castellanos (2007)) then $k^* = 7.2649$**.**

The control limits as well as the means and 95% Bayesian confidence intervals for e_{ij}^* are given in Figure 6.4, and in Figure 6.5, the unconditional posterior distributions are illustrated. As mentioned, the unconditional posterior distributions are obtained using the Rao-Blackwell method.

Figure 6.4: Means and 95% Intervals for $e_{ij}^*, i = 1, \dots, 5,\ j = 1, \dots, 6$



Standardised Errors

Figure 6.5: Unconditional Posteriors of $e_{41}^*, e_{53}^*, e_{16}^*$

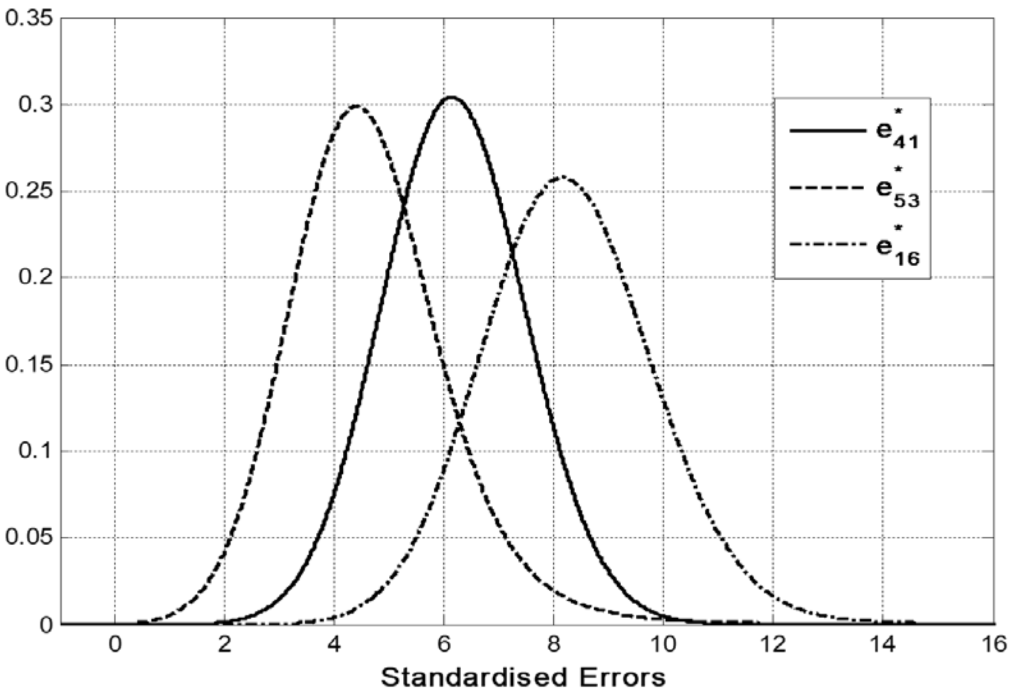


Table 6.1 Probabilities that Individual Observations are Outlying

e_{ij}^*	$Mean(e_{ij}^*)$	$Var(e_{ij}^*)$	95% Interval	$P(\lvert e_{ij}^*\rvert > 7.26)$
41	6.1981	1.7257	3.623 – 8.773	0.2094
53	4.6637	2.2356	1.773 – 7.954	0.0412
16	8.3100	2.5608	5.174 – 11.447	0.7451

Since the mean of e_{16}^* is larger than $k^* = 7.26$, it can be concluded that Y_{16} is an outlying observation. This is also clear from Table 6.1 where $P(|e_{16}^*| > 7.26) = 0.7451$.

If the null hypothesis is true, only 5% of $max|Y_i^*|$ will be larger than $\tilde{k} = 3.44$ and only 5% of $max|Y_{ij}^*|$ will be larger than $\tilde{k} = 7.2649$. If these values of $\tilde{k}$ are used, the Type I error rate will be 5%. These results seem to us an improvement on Chaloner's (1994) and Zellner's (1975) results. We however did not investigate the power of the tests.

7. Conclusion

In this note the Bayesian procedures of Zellner (1975) and Chaloner (1994) are extended for the balanced one-way random effects model. Since it is not clear to us what the frequentist properties of their methods are (i.e. what the size of the Type I error or the power of their tests are), a Bayesian-frequentist approach is used for detecting outliers. The Sharples generated data (Sharples (1990) and Chaloner (1994)) are used for illustration purposes.

Chaloner (1994) concluded that observations Y_{16} and Y_{41} as well as Group 5 are possible outliers, since their posterior probabilities are larger than the prior probability of an individual observation or group being an outlier.

Inspection of Tables 5.3 and 6.1, as well as Figures 5.3 and 6.2 show that only observation Y_{16} is an outlier.

The Bayesian-frequentist approach therefore seems to be more conservative than Chaloner's method.

Appendix A

Proof of Theorem 5.1

$$Y_i^* = \frac{J\sigma_2\sqrt{I}(\bar{Y}_{i.}-\bar{Y}_{..})}{\sqrt{(\sigma_1^2+J\sigma_2^2)(I\sigma_1^2+J\sigma_2^2)}} \quad i = 1, \dots, I$$

where

$$\bar{Y}_{i.} = \frac{1}{J}\sum_{j=1}^{J} Y_{ij} = \frac{1}{J}\sum_{j=1}^{J}(\theta + r_i + e_{ij}) = \theta + r_i + \bar{e}_{i.}$$

and

$$\bar{Y}_{..} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} Y_{ij} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}(\theta + r_i + e_{ij}) = \theta + \bar{r}_{.} + \bar{e}_{..}$$

Therefore

$$E(\bar{Y}_{i.} - \bar{Y}_{..}) = \theta - \theta = 0$$

Since r_i and e_{ij} are uncorrelated it follows that

$$\begin{aligned} Var(\bar{Y}_{i.} - \bar{Y}_{..}) &= Var(r_i - \bar{r}_{.}) + Var(\bar{e}_{i.} - \bar{e}_{..}) \\ &= \sigma_2^2\frac{(I-1)}{I} + \sigma_1^2\frac{(I-1)}{IJ} \\ &= \frac{(I-1)}{IJ}(\sigma_1^2 + J\sigma_2^2) \quad i = 1, \dots, I \end{aligned}$$

and therefore

$$Var(Y_i^*) = \frac{J(I-1)\sigma_2^2}{(I\sigma_1^2+J\sigma_2^2)} \tag{A.1}$$

Also

$$\begin{aligned} Cov(\bar{Y}_{l.} - \bar{Y}_{..})(\bar{Y}_{m.} - \bar{Y}_{..}) &= E(\bar{Y}_{l.} - \bar{Y}_{..})(\bar{Y}_{m.} - \bar{Y}_{..}) - E(\bar{Y}_{l.} - \bar{Y}_{..})E(\bar{Y}_{m.} - \bar{Y}_{..}) \\ &= E(\bar{Y}_{l.} - \bar{Y}_{..})(\bar{Y}_{m.} - \bar{Y}_{..}) \\ &= E\{[(r_l - \bar{r}_{.}) + (\bar{e}_{l.} - \bar{e}_{..})][(r_m - \bar{r}_{.}) + (\bar{e}_{m.} - \bar{e}_{..})]\} \\ &= E[(r_l - \bar{r}_{.})(r_m - \bar{r}_{.})] + E[(\bar{e}_{l.} - \bar{e}_{..})(\bar{e}_{m.} - \bar{e}_{..})] \end{aligned}$$

since the expected values of the cross products are zero.

Now $E[(r_l - \bar{r}_.)(r_m - \bar{r}_.)] = E[r_l r_m] - E[r_l \bar{r}_.] - E[\bar{r}_. r_m] + E[\bar{r}_.^2]$

$$= 0 - \frac{\sigma_2^2}{I} - \frac{\sigma_2^2}{I} + \frac{\sigma_2^2}{I}$$

$$= \frac{-\sigma_2^2}{I}$$

and

$E[(\bar{e}_{l.} - \bar{e}_{..})(\bar{e}_{m.} - \bar{e}_{..})] = E[\bar{e}_{l.}\bar{e}_{m.}] - E[\bar{e}_{l.}\bar{e}_{..}] - E[\bar{e}_{..}\bar{e}_{m.}] + E[\bar{e}_{..}^2]$

$$= 0 - \frac{\sigma_1^2}{IJ} - \frac{\sigma_1^2}{IJ} + \frac{\sigma_1^2}{IJ}$$

$$= \frac{-\sigma_1^2}{IJ}$$

Therefore

$$Cov(\bar{Y}_{l.} - \bar{Y}_{..})(\bar{Y}_{m.} - \bar{Y}_{..}) = \frac{-1}{IJ}(\sigma_1^2 + J\sigma_2^2)$$

and

$$Cov(Y_l^*, Y_m^*) = \frac{-J\sigma_2^2}{(I\sigma_1^2 + J\sigma_2^2)}, l = 1, \dots, I,\ m = 1, \dots, I, l \neq m.$$

From this it follows that the correlation coefficient between

Y_l^* and $Y_m^* = \rho_{l,m} = \frac{-1}{I-1}$ (A.2)

Appendix B

Proof of Theorem 6.1

$$Y_{ij}^* = \frac{Y_{ij} - \frac{J\sigma_2^2}{\sigma_1^2 + J\sigma_2^2}\bar{Y}_{i.} - \frac{\sigma_1^2}{\sigma_1^2 + J\sigma_2^2}\bar{Y}_{..}}{\sqrt{\frac{\sigma_1^2}{\sigma_1^2 + J\sigma_2^2}\left(\sigma_2^2 + \frac{\sigma_1^2}{IJ}\right)}},\ i = 1, \dots, I,\ j = 1, \dots, J$$

Let $\tilde{Y}_{ij} = Y_{ij} - \frac{J\sigma_2^2}{\sigma_1^2 + J\sigma_2^2}\bar{Y}_{i.} - \frac{\sigma_1^2}{\sigma_1^2 + J\sigma_2^2}\bar{Y}_{..}$

$$= Y_{ij} - a\bar{Y}_{i.} - b\bar{Y}_{..}$$

where $a = \frac{J\sigma_2^2}{\sigma_1^2 + J\sigma_2^2}$, $b = \frac{\sigma_1^2}{\sigma_1^2 + J\sigma_2^2}$ and $a + b = 1$

The model is

$Y_{ij} = \theta + r_i + e_{ij}$

where $e_{ij} \sim N(0, \sigma_1^2)$ and $r_i \sim N(0, \sigma_2^2)$

(a) we will first derive $Var(Y_{ij}^*)$

Now $Var(\tilde{Y}_{ij}) = Var(r_i - ar_i - b\overline{r}_.) + Var(e_{ij} - a\overline{e}_{i.} - b\overline{e}_{..})$

since r_i and e_{ij} are independently distributed of each other

Further

$Var(r_i - ar_i - b\overline{r}_.) = Var[b(r_i - \overline{r}_.)] = b^2\sigma_2^2\frac{(I-1)}{I}$

and

$Var(e_{ij} - a\overline{e}_{i.} - b\overline{e}_{..}) = \frac{\sigma_1^2}{J}\left[(J-1) + b^2\left(\frac{I-1}{I}\right)\right]$

Therefore

$Var(\tilde{Y}_{ij}) = \frac{\sigma_1^2}{J}\left\{(J-1) + \frac{\sigma_1^2}{\sigma_1^2 + J\sigma_2^2}\left(\frac{I-1}{I}\right)\right\}$

and

$$Var(Y_{ij}^*) = \frac{1}{\sigma_1^2 + IJ\sigma_2^2}\{I(J-1)(\sigma_1^2 + J\sigma_2^2) + \sigma_1^2(I-1)\} \quad \text{(B.1)}$$

(b) $Cov(\tilde{Y}_{ij}, \tilde{Y}_{lm}) = E(\tilde{Y}_{ij}\tilde{Y}_{lm})$ because $E(\tilde{Y}_{ij}) = E(\tilde{Y}_{lm}) = 0$

$(i = 1, \ldots, I;\ j = 1, \ldots, J; l = 1, \ldots, I;\ m = 1, \ldots, J; i \neq l \text{ and } j \neq m)$

Therefore

$$\begin{aligned} Cov(\tilde{Y}_{ij}, \tilde{Y}_{lm}) &= E(Y_{ij} - a\overline{Y}_{i.} - b\overline{Y}_{..})(Y_{lm} - a\overline{Y}_{l.} - b\overline{Y}_{..}) \\ &= E\{b(r_i - \overline{r}_.) + (e_{ij} - a\overline{e}_{i.} - b\overline{e}_{..})\}\{b(r_l - \overline{r}_.) + (e_{lm} - a\overline{e}_{l.} - b\overline{e}_{..})\} \\ &= b^2E(r_i - \overline{r}_.)(r_l - \overline{r}_.) + E(e_{ij} - a\overline{e}_{i.} - b\overline{e}_{..})(e_{lm} - a\overline{e}_{l.} - b\overline{e}_{..}) \end{aligned}$$

Now

$b^2E(r_i - \overline{r}_.)(r_l - \overline{r}_.) = \frac{-b^2\sigma_2^2}{I}$

and

$E(e_{ij} - a\overline{e}_{i.} - b\overline{e}_{..})(e_{lm} - a\overline{e}_{l.} - b\overline{e}_{..}) = \frac{-b^2\sigma_1^2}{IJ}$

From this it follows that

$Cov(\tilde{Y}_{ij}, \tilde{Y}_{lm}) = \frac{-b^2}{IJ}(\sigma_1^2 + J\sigma_2^2) = \frac{-(\sigma_1^2)^2}{IJ(\sigma_1^2 + J\sigma_2^2)}$

and

$$Cov(Y_{ij}^*, Y_{lm}^*) = \frac{-\sigma_1^2}{\sigma_1^2 + IJ\sigma_2^2} \quad \text{(B.2)}$$

(c) In a similar way it can be shown that

$$Cov(Y_{ij}^*, Y_{lj}^*) = \frac{-\sigma_1^2}{\sigma_1^2 + IJ\sigma_2^2} \quad \text{(B.3)}$$

(d) $Cov(\tilde{Y}_{ij}, \tilde{Y}_{im}) = b^2 E(r_i - \overline{r}_.)^2 + E(e_{ij} - a\overline{e}_{i.} - b\overline{e}_{..})(e_{im} - a\overline{e}_{i.} - b\overline{e}_{..})$

$= b^2\sigma_2^2 \frac{(I-1)}{I} - \frac{\sigma_1^2}{J}\left\{(1-b^2) + \frac{b^2}{I}\right\} = -\frac{\sigma_1^2}{J}\left\{\frac{\sigma_1^2 + IJ\sigma_2^2}{I(\sigma_1^2 + J\sigma_2^2)}\right\}$

and $Cov(Y_{ij}^*, Y_{im}^*) = -1$ (B.4)

Appendix C

Proof of Theorem 6.2

We show here that the integrals (on the interval $[0,\infty)$) of 6.3 and 6.4 are convergent. First, a few inequalities (valid for all $\gamma > 0$):

$P_{R2}(\gamma) = \left(\sum_{i=1}^{I} \frac{{n_i}^2}{(1+\gamma[\![n_i)]\!]^2}\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{I} \frac{{n_i}^2}{\gamma^2 {n_i}^2}\right)^{\frac{1}{2}} = \frac{\sqrt{I}}{\gamma}.$

$\prod_{i=1}^{I}\left(\frac{1}{1+\gamma n_i}\right)^{\frac{1}{2}} \le \prod_{i=1}^{I}\left(\frac{1}{\gamma n_i}\right)^{\frac{1}{2}} \le \prod_{i=1}^{I}\left(\frac{1}{\gamma n\prime}\right)^{\frac{1}{2}} = K_1\left(\frac{1}{\gamma}\right)^{\frac{I}{2}},$ for a constant $K_1 > 0$.

where $n' = \min_{1\le i\le I} n_i$

$\left(\sum_{i=1}^{I} \frac{n_i}{1+\gamma n_i}\right)^{\frac{1}{2}} \ge \left(\sum_{i=1}^{I} \frac{n'}{1+\gamma n"}\right)^{\frac{1}{2}} = \sqrt{\frac{I\cdot n\prime}{1+\gamma n"}},$ where $n" = \max_{1\le i\le I} n_i$

implying that $\left(\sum_{i=1}^{I} \frac{n_i}{1+\gamma n_i}\right)^{-\frac{1}{2}} \le \frac{1}{\sqrt{I\cdot n'}}\left(\sqrt{1+\gamma n"}\right) \le K_2\sqrt{1+\gamma},$ for a constant $K_2 > 0$.

In the event that $\overline{Y}_{1.} = \overline{Y}_{2.} = \cdots = \overline{Y}_{I.}$, we have that $\hat{\theta}(\gamma) = \overline{Y}_{1.} = \overline{Y}_{2.} = \cdots = \overline{Y}_{I.}$ for all $\gamma > 0$, whence the expression inside the square brackets of equation (6.4) reduces to $v_1 m_1$. Otherwise, there is a non-empty subset J of {1,2,…,I} such that $\overline{Y}_{i.}$ is different from $\overline{Y} = \frac{1}{I}\sum_{i=1}^{I}\overline{Y}_{i.}$ for all $i \in J$. Select $t \in J$ such that $\overline{Y}_{t.} < \overline{Y}$ and $\overline{Y} - \overline{Y}_{t.}$ is as small as possible. Similarly, select $s \in J$ such that $\overline{Y}_{s.} > \overline{Y}$ and $\overline{Y}_{s.} - \overline{Y}$ is as small as possible. By a straightforward application of L'Hospital's Rule, we find that $\lim_{\gamma\to\infty}\hat{\theta}(\gamma) = \bar{Y}$. Put $\tau_1 = \frac{1}{2}(\overline{Y} + \overline{Y}_{t.})$ and $\tau_2 = \frac{1}{2}(\overline{Y} + \overline{Y}_{s.})$. Then there exists $\gamma_0 > 0$ such that $\tau_1 < \hat{\theta}(\gamma) < \tau_2$ for all $\gamma > \gamma_0$. It follows that, with $M' = \min\{(Y-\tau_1)^2, \left(\tau_2 - \overline{Y}\right)^2\}$ that $(\hat{\theta}(\gamma) - \bar{Y}_{i.})^2 > M'$ for all $\gamma > \gamma_0$ and all $i \in J$. Consequently, $\sum_{i=1}^{I}\frac{n_i\left(\overline{Y}_{i.} - \hat{\theta}\right)^2}{1+\gamma n_i} \ge M'\cdot n'\cdot\frac{|J|}{1+\gamma n"}$ for all $\gamma > \gamma_0$. Hence,

$$\left[v_1 m_1 + \sum_{i=1}^{I}\frac{n_i\left(\overline{Y}_{i.} - \hat{\theta}\right)^2}{1+\gamma n_i}\right]^{-\frac{1}{2}(n-1)} \le \left[\frac{|J|\cdot M'\cdot n'}{1+\gamma n"}\right]^{-\frac{1}{2}(n-1)} = K_3\cdot\frac{1}{(1+\gamma n")^{(n-1)/2}},$$

for a constant $K_3 > 0$, and all $\gamma > \gamma_0$.

From the foregoing, it follows that, for $\gamma > \gamma_0$,

$$0 \leq P_{R2}(\gamma) \cdot \prod_{i=1}^{I} \left(\frac{1}{1+\gamma n_i}\right)^{\frac{1}{2}} \cdot \left[v_1 m_1 + \sum_{i=1}^{I} \frac{n_i(\overline{Y}_{i.}-\hat{\theta})^2}{1+\gamma n_i}\right]^{-\frac{1}{2}(n-1)}$$

$$\leq K \cdot \frac{\sqrt{I}}{\gamma} \cdot \frac{1}{\gamma^{\frac{I}{2}}} \cdot \sqrt{1+\gamma} \cdot \frac{1}{(1+\gamma n")^{\frac{n-1}{2}}} = K \cdot \omega(\gamma),$$ say, for a constant $K > 0$.

It is straightforward to check that $\lim_{\gamma\to\infty} \gamma^2 \omega(\gamma) = 0$, even for the special case $\overline{Y}_{1.} = \overline{Y}_{2.} = \cdots = \overline{Y}_{I.}$. Hence $0 \leq K \cdot \omega(\gamma) \leq \frac{K}{\gamma^2}$ for all $\gamma > \gamma_1$ (say). It follows that the improper integral

$$\int_0^{\infty} P_{R2}(\gamma) \cdot \prod_{i=1}^{I} \left(\frac{1}{1+\gamma n_i}\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{I} \frac{n_i}{1+\gamma n_i}\right)^{-\frac{1}{2}} \cdot \left[v_1 m_1 + \sum_{i=1}^{I} \frac{n_i(\overline{Y}_{i.}-\hat{\theta})^2}{1+\gamma n_i}\right]^{-\frac{1}{2}(n-1)} d\gamma$$

is convergent, as it is well-known that $\int_{\gamma_2}^{\infty} \frac{1}{\gamma^2} d\gamma$ is convergent (where we take $\gamma_2 = \max\{\gamma_0, \gamma_1\}$).

Since $0 \leq P_{R1}(\gamma) \leq P_{R2}(\gamma)$ for all $\gamma > 0$, we immediately also have that

$$\int_0^{\infty} P_{R1}(\gamma) \cdot \prod_{i=1}^{I} \left(\frac{1}{1+\gamma n_i}\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{I} \frac{n_i}{1+\gamma n_i}\right)^{-\frac{1}{2}} \cdot \left[v_1 m_1 + \sum_{i=1}^{I} \frac{n_i(\overline{Y}_{i.}-\hat{\theta})^2}{1+\gamma n_i}\right]^{-\frac{1}{2}(n-1)} d\gamma$$

is convergent.

Appendix D

Table D1: Differences between $\mathbf{P_{R1}}(\gamma|\underline{Y})$ and $\mathbf{P_{R2}}(\gamma|\underline{Y})$

γ	**0.4**	**0.8**	**1.2**	**1.6**	**2**
$\{\Pr_1(\gamma\|\underline{Y}) - \Pr_2(\gamma\|\underline{Y})\}10^7$	29.5684	31.577	17.609	6.177	-0.489

γ	**10**	**20**	**30**	**40**	**50**
$\{\Pr_1(\gamma\|\underline{Y}) - \Pr_2(\gamma\|\underline{Y})\}10^7$	-1.2335	-0.2306	-0.07803	-0.035171	-0.018733

γ	**60**	**70**	**80**	**90**	**100**
$\{\Pr_1(\gamma\|\underline{Y}) - \Pr_2(\gamma\|\underline{Y})\}10^7$	-0.011131	-0.007143	-0.004853	-0.003346	-0.002534

The differences between the two posterior distributions are multiplied by 10^7. It is clear that they are for all practical purposes zero.

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